MATH 504 SPRING TERM 2009, SOLUTIONS 5

1. Evaluate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}.$$

By Problems 3, when $0 \le x \le 1$, $B_3(x) = -\sum_{n\ne 0} \frac{e(nx)}{(2\pi i n)^3}$. Moreover $\sum_{n\ne 0} \frac{e(n/4)}{n^3} = \sum_{n=1}^{\infty} \frac{i^n - i^{-n}}{n^3}$ and $i^n - i^{-n} = 0$ when n is even, is $2i$ when $n \equiv 1 \pmod 4$ and is $-2i$ when $n \equiv 3 \pmod 4$. Thus it is $2i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}$. On the other hand $B_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x$ and so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = -i^2 4\pi^3 B_3(1/4) = \frac{\pi^3}{32}$.

2. Find a Fourier series proof that if m and n are non–negative integers, then

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

We have $\binom{k}{m} = \int_0^1 \sum_{r=0}^{k} \binom{k}{r} e(r\alpha)e(-m\alpha)d\alpha = \int_0^1 (1+e(\alpha))^k e(-m\alpha)d\alpha$. Hence $\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \int_0^1 \sum_{k=0}^{n} \binom{n}{k}(1+e(\alpha))^k e(-m\alpha)d\alpha = \int_0^1 (2+e(\alpha))^n e(-m\alpha)d\alpha = 2^{n-m}\binom{n}{m}$.

3. Suppose that $g : [0,1] \to \mathbb{C}$, $g \in L^2[0,1]$, $c \in \mathbb{C}$, $f(x) = \int_0^x g(y)dy + c$ $(0 \le x \le 1)$, $c_0(f) = \int_0^1 f(x)dx$, $c_0(g) = \int_0^1 g(x)dx$. Then prove that

$$\int_0^1 \left| f(x) - \big(f(1) - f(0)\big)(x - \tfrac{1}{2}) - c_0(f) \right|^2 dx \le \frac{1}{4\pi^2} \int_0^1 |g(x) - c_0(g)|^2 dx$$

with equality if and only if f is of the form $ax + b + ce^{2\pi i x} + de^{-2\pi i x}$.

When $k \ne 0$ we have $\hat{f}(k) = \int_0^1 e(-kx) \int_0^x g(y)dydx = \int_0^1 g(y) \int_y^1 e(-kx)dxdy = \int_0^1 g(y) \frac{e(-ky)-1}{2\pi i k} dy = \frac{\hat{g}(k)-1}{2\pi i k}$. Moreover $\hat{f}(0) = \int_0^1 f(x)dx = c_0(f)$, $f(1) - f(0) = \int_0^1 g(x)dx = c_0(g)$ and $\hat{g}(0) = c_0(g)$. Thus $\sum_k \hat{f}(k)e(kx) = \sum_{k\ne 0} \frac{\hat{g}(k)}{2\pi i k} e(kx) + c_0(f) + \sum_{k \ne 0} \frac{-c_0(g)}{2\pi i k} e(kx)$ and from homework 3 the last series is $c_0(g)(x - \frac{1}{2}) = (f(1) - f(0))(x - \frac{1}{2})$ when $0 < x < 1$. Hence, by Parseval's indentity, $\int_0^1 \big| f(x) - \big(f(1)-f(0)\big)(x-\frac{1}{2}) - c_0(f) \big|^2 dx = \sum_{k\ne 0} \left| \frac{\hat{g}(k)}{2\pi i k} \right|^2 \le \frac{1}{4\pi^2} \sum_{k \ne 0} |\hat{g}(k)|^2 = \frac{1}{4\pi^2} \int_0^1 |g(x) - \hat{g}(0)|^2 dx$. Finally we have equality only when $\hat{g}(k) = 0$ when $|k| \ge 2$. Thus in the extremal case g has the fourier expansion $c_{-1}e(-x) + c_0 + c_1 e(x)$ and so $f(x) = \int_0^x \big(c_{-1}e(-y) + c_0 + c_1 e(y)\big)dy + c$.