MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, SOLUTIONS 4

1. §1.5 Exercise 1. Show that there are α, β in $L^1(S^1)$ such that $\|\alpha+\beta\|_1^2 + \|\alpha-\beta\|_1^2 \neq 2\|\alpha\|_1^2 + 2\|\beta\|_1^2$.

Let $\alpha(x) = \frac{1}{2}$, $\beta(x) = x$. Then $\|\alpha+\beta_1| = \int_0^1 (\frac{1}{2}+x)dx = 1$, $\|\alpha-\beta\| = \int_0^1 |\frac{1}{2}-x|dx = \int_0^{1/2}(\frac{1}{2}-x)dx + \int_{1/2}^1 (x-\frac{1}{2})dx = \frac{1}{4}$, so LHS$= \frac{17}{16}$. Also $\|\alpha\|_1 = \frac{1}{2}$ and $\|\beta\|_1 = \frac{1}{2}$, so RHS$= 1$.

2. §1.5 Exercise 8. Prove that if $f \in L^1(S^1)$ and if for each fixed $x \in S^1$ the function $g_x(y) = y^{-1}\big(f(x+y) - f(x)\big)$ is also in $L^1(S^1)$, then $S_n(x) = \sum_{|k|\leq n} \hat{f}(k)e_k(x)$ converges to f. Hint: Use the Dirichlet kernel and the Riemann–Lebesgue lemma.

As usual, $\sum_{|k|\leq n} \hat{f}(k)e_k(x) - f(x) = \int_{-1/2}^{1/2} \big(f(x+y)-f(x)\big)D_n(y-x)dy$. The integrand is $g_x(y)\frac{y}{\sin \pi y}\sin \pi(2n+1)y$. The function $\frac{y}{\sin \pi y}$ $(y \neq 0)$, $1/\pi$ $(y=0)$ is continuous on $[-\frac{1}{2}, \frac{1}{2}]$, so the integrand is of the form $h(y)\sin \pi(2n+1)y$ where $h \in L^1(S^1)$. Hence, by the Riemann–Lebesgue lemma, $\sum_{|k|\leq n} \hat{f}(k)e_k(x) - f(x) \to 0$ as $n \to \infty$.

3. §1.5 Exercise 12. Prove that the operation of convolution on $L^1(S^1)$ does not have an identity element. Hint: The Riemann–Lebesgue Lemma is useful here.

Suppose on the contrary that there is an identity g, i.e. there is a $g \in L^1(S^1)$ such that for every $f \in L^1(S^1)$ we have $f \circ g = f$. Then for every $k \in \mathbb{Z}$, $\hat{f}(k) = \widehat{f \circ g}(k) = \hat{f}(k)\hat{g}(k)$. But $\hat{g}(k) \to 0$ as $|k| \to \infty$. Hence there is a k_0 such that if $|k| > k_0$, then $|\hat{g}(k)| < \frac{1}{2}$, and so $|\hat{f}(k)| \leq \frac{1}{2}|\hat{f}(k)|$, whence $\hat{f}(k) = 0$. Obviously there are $f \in L^1(S^1)$ for which this is false.