Math 504 Spring Term 2009, Solutions 3

1. Let the polynomials $B_p(x)$ for $k \in \mathbb{N}$ be defined on $\mathbb{R}$ by $B_1(x) = x - \frac{1}{2}$, $B_{p+1}(x) = \int_0^x B_p(y)dy - \int_0^1 (1-y)B_p(y)dy$ and let $\hat{B}_p(k)$ denote the Fourier coefficient (relative to the family $e(kx)$ on $L^2(S_1)$). (i) Prove that for $p > 1$ we have $B_p(1) = B_p(0)$ and deduce that B_p restricted to $\mathbb{R}/\mathbb{Z}$ is continuous. (ii) Prove that $\hat{B}_p(0) = 0$ and $\hat{B}_p(k) = -(2\pi i k)^{-p}$ when $k \in \mathbb{Z}\backslash\{0\}$. (iii) Prove that when p is even, then $\sum_{k=1}^{\infty} k^{-p} = 2^{p-1}\pi^p(-1)^{p/2-1}B_p(0)$. (iv) Prove that $B_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$, $B_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}$ and that generally B_p is a polynomial of degree p with rational coefficients. (v) Prove that $\zeta(2) = \frac{\pi^2}{6}$ and that $\zeta(4) = \frac{\pi^4}{90}$.

(i) We have $\int_0^1 B_1(y)dy = 0$ and for $p > 1$ we have $\int_0^1 B_p(y)dy = \int_0^1 \left(\int_0^x B_{p-1}(y)dx\right)dy - \int_0^1 (1-y)B_p(y)dy$. On interchanging the order of integration in the repeated integral it follows that it equals the last integral. Hence for every p, $\int_0^1 B_p(y)dy = 0$. But for $p > 1$, $B_p(1) - B_p(0) = \int_0^1 B_{p-1}(y)dy$. (ii) We have $\hat{B}_p(0) = \int_0^1 B_p(y)dy = 0$. When $p > 1$ integration by parts gives for $k \neq 0$, $\hat{B}_1(k) = -(2\pi i k)^{-1}$ and $\hat{B}_p(k) = (2\pi i k)^{-1}\hat{B}_{p-1}(k)$. (iii) For p even, $\sum_{k=1}^{\infty} k^{-p} = \frac{1}{2}\sum_{k\neq 0} k^{-p} = 2^{p-1}(\pi i)^p \sum_{k\neq 0}(2\pi i k)^{-p} = 2^{p-1}\pi^p(-1)^{p/2}\sum_k \hat{B}_p(k)$. Moreover $B_p \in \mathcal{C}^p(S^1)$ and so the Fourier series converges to B_p. (iv) This is a routine calculation using the iterative definition. As a check $B_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{2}x$. (v) $\zeta(2) = 2\pi^2(-1)^0/12 = \pi^2/6$. $\zeta(4) = 2^3\pi^4(-1)(-1/720) = \pi^4/90$.

2. In the notation of the previous question, B_1 is the most interesting of the functions since it is not continuous on $\mathbb{R}/\mathbb{Z}$. It has a jump discontinuity at 0. Apart from the redefinition at 0 it coincides on $[0,1)$ with the sawtooth function $s(x)$ defined by $s(0) = 0$, $s(x) = x - \frac{1}{2}$ when $0 < x < 1$ and otherwise by periodicity with period 1. Let $E_K(x) = s(x) + \sum_{0<|k|\leq K} \frac{e(kx)}{2\pi i k}$. (i) Prove that $E_k(x)$ is an odd function of x. (ii) Prove that if $x \notin \mathbb{Z}$, then $E_K'(x) = D_K(x)$ (the Dirichlet kernel). (iii) Prove that if $0 < x < 1$, then $E_k(x) = E_k(x) - E_k(\frac{1}{2}) = \int_{\frac{1}{2}}^x D_K(y)dy$. (iv) Prove that $\int_{\frac{1}{2}}^x D_K(y)dy = \left[\frac{1-\cos(\pi(2K+1)y)}{(2K+1)\pi\sin\pi y}\right]_{1/2}^x + \int_{1/2}^x \frac{1-\cos(\pi(2K+1)y)}{(2K+1)\sin^2\pi y}\cos\pi y dy$. (v) Prove that if $0 < x < 1$, then $|E_k(x)| \leq \frac{2}{(2K+1)\pi\sin\pi x}$. (vi) Prove that for all x, $|E_k(x)| \leq \frac{1}{2}$. The facts that E_k is odd and that when $0 < x \leq \frac{1}{2}$ we have $2x \leq \sin\pi x \leq \pi x$ are useful here. (vii) Prove that $-\sum_{0<|k|\leq K} \frac{e(kx)}{2\pi i k}$ converges to $s(x)$. (viii) Prove that $\|E_K\|_2 = O(K^{-1/2})$ (relative to $L^2(\mathbb{R}/\mathbb{Z})$).

(i) $E_K(-x) = E_K(1-x) = \frac{1}{2} - x + \sum_{0<|k|\leq K} e(-kx)/(2\pi i k) = -E_k(x)$. (ii) $E_K'(x) = 1 + \sum_{O<|k|\leq K} e(kx) = D_K(x)$. (iii) $E_k(x) - E_k(1/2) = \int_{1/2}^x E_K'(y)dy$. (iv) Use $D_k(y) = \frac{\sin\pi(2K+1)y}{\sin\pi y}$ and integration by parts. (v) We have $0 \leq 1 - \cos(\pi(2K+1)y) \leq 1$, $0 < \sin\pi y < 1$, $\sin\pi/2 = 1$ and $\cos\pi y$ is of constant sign between $1/2$ and x. Thus the first term lies between $\pm\frac{1}{(2K+1)\pi\sin\pi x}$ and the second is bounded by $\pm\frac{1}{2K+1}\int_{1/2} \frac{\cos\pi y}{(\sin\pi y)^2}dy$ and this in turn is bounded by $\pm\frac{1}{(2K+1)\pi\sin\pi x}$. (vi) $E_k(0) = 0$ and E_K is odd so we can suppose that $0 < x \leq 1/2$. By (v) $|E_K(x)| \leq \frac{1}{(2K+1)\pi x}$ so we can suppose that $0 < x \leq x_0$ where $x_0 = \frac{2}{(2K+1)\pi}$, and $|E_K(x_0)| \leq 1/2$. Now $E_K'(x) = \frac{\sin\pi(2K+1)x}{\sin\pi x}$ and $\pi(2K+1)x < \pi$ so $E_K'(x) > 0$. Moroever $\lim_{x\to 0+} E_K(x) = -1/2$. Hence by the mean value theorem $E_K(x) > -1/2$ and $E_K(x) \leq E_K(x_0)$ on $(0, x_0]$. But $E_K(x_0) \leq |E_K(x_0)| \leq 1/2$. (vii) Trivial when $x \in \mathbb{Z}$ and immediately from (v) when $x \notin \mathbb{Z}$. (viii) Apply (v) and (vi). Then $\|E_K\|_2^2 \leq 2\int_0^{1/2} \min\left(1/4, 1/\left((2K+1)^2\pi^2x^2\right)\right)dx = O(K^{-1})$.