MATH 504 ANALYSIS IN EUCLIDEAN SPACES, SPRING TERM 2009, SOLUTIONS 2

1. §1.4. Exercise 5. Prove that if $\{z_n\}$ is a sequence of complex numbers such that $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} \frac{1}{n}(z_1 + \cdots + z_n) = z$. Give an example in which the second limit exists but the first does not.

Let $l = \lim z_n$ and $\varepsilon > 0$. Choose M so that for every $m > M$ we have $|z_m - l| < \varepsilon/2$ and choose $N > M$ so that $|z_1 + \cdots + z_M - lM|/N < \varepsilon/2$. Then, whenever $n > N$ we have $|(z_1+\cdots+z_n)/n - l| \le |z_1 + \cdots + z_M - Ml|/n + |(z_{M+1} - l) + \cdots + (z_n - l)|/n < \varepsilon/2 + (|z_{M+1} - l| + \cdots + |z_n - l|)/n < \varepsilon/2 + (n-M)\epsilon/2n < \varepsilon$.

$z_n = (-1)^n$ diverges but then $(z_1 + \cdots + z_n)/n = \frac{(-1)^n - 1}{2n}$ converges.

2. §1.4. Exercise 8. Suppose that $f \in \mathcal{C}(S^1)$ and $0 \le r < 1$. Prove that

$$\sum_n \hat{f}(n) r^{|n|} e(nx) = \int_0^1 \frac{1 - r^2}{1 - 2r\cos(2\pi(x-y)) + r^2} f(y)dy.$$

Since $f \in \mathcal{C}(S^1)$, f is uniformly bounded on S^1. Hence by the Weierstrasse M–test $\sum_n f(y) r^{|n|} e(-ny)e(nx)$ converges uniformly for $y \in S^1$. Hence

$$\int_0^1 \sum_n f(y) r^{|n|} e(-ny) e(nx) dy = \sum_n \hat{f}(n) r^{|n|} e(nx).$$

The integrand is

$$f(y)\left(1 + \sum_{n=1}^{\infty} \big(e(n(x-y)) + e(n(y-x))\big) r^n\right)$$

and it is readily checked that this sums to the integrand in question.

3. §1.5. Exercises 4 and 5. Find a function f in $L^1[0,1]$ which is not in $L^2[0,1]$, and a function g in $L^2(\mathbb{R})$ which is not in $L^1(\mathbb{R})$.

$|x - 0.5|^{-1/2} \in L^1[0,1]$, $\notin L^2[0,1]$.
$(1 + |x|)^{-3/4} \in L^2(\mathbb{R})$, $\notin L^1(\mathbb{R})$.