Math 504 Spring Term 2009, Solutions 1

1. §1.1. Exercise 2. Prove that if $f_1, f_2 \ldots$ are real continuous functions of $\mathbb{R}$ and if for each $x \in \mathbb{R}$ we have $f(x) \lim_{n\to\infty} f_n(x)$ exists, then $\{x : 0 \le f(x) < 1\}$ is measurable. Hint: Prove that $\{x : f(x) < 1\} = \cup_{k\ge1} \cup_{m\ge1} \cap_{n\ge m} \{x : f_n(x) \le 1 - 1/k\}$.

I made some corrections above. Let $\mathcal{A}_\lambda = \{x : f(x) < \lambda\}$, $\mathcal{B}_\lambda = \cup_{k\ge1} \cup_{m\ge1} \cap_{n\ge m} \{x : f_n(x) \le \lambda - 1/k\}$, where $\lambda \in \mathbb{R}$. As the f_n are continuous, $\mathcal{B}_\lambda$ is a countable union of a countable intersection of a countable union of intervals so is measurable. Thus it suffices to show that $\mathcal{B}_\lambda = \mathcal{A}_\lambda$ because $\{x : 0 \le f(x) < 1\} = \mathcal{A}_1 \backslash \mathcal{A}_0$. Let $x \in \mathcal{A}_\lambda$. Then $f(x) < \lambda$. Choose k so that $f(x) < \lambda - 2/k$. Then there is an $m = m(x, k)$ so that whenever $n \ge m$ we have $f_n(x) < \lambda - 1/k$ and so $x \in \mathcal{B}_\lambda$. Now suppose that $x \in \mathcal{B}_\lambda$. Then there are k, m such that $f_n(x) \le \lambda - 1/k$ whenever $n \ge m$. By the existence of $f(x)$ there is an $n_0 = n_0(x, k)$ such that whenever $n > n_0$ we have $|f_n(x) - f(x)| < 1/k$. Thus $f(x) < \lambda$ and so $x \in \mathcal{A}_\lambda$.

2. §1.2. Exercise 2. Check for fixed β that the inner product (α, β) is continuous in α.

Here we are supposing that the α, β are members of $L^2(Q)$. Thus continuity is the statement that if ϕ is some functional on the space, then ϕ is continuous on the space provided that $\phi(\alpha) \to \phi(\alpha_0)$ as $\alpha \to \alpha_0$. Moreover the measure of "size" here will be $\|.\|_2$. Now by Schwarz's inequality, $|(\alpha, \beta) - (\alpha_0, \beta)| = |(\alpha - \alpha_0, \beta)| \le \|\alpha - \alpha_0\|_2 . \|\beta\|_2 \to 0$ as $\alpha \to \alpha_0$.

3. §1.3. Exercise 9. For $x \in [0, 1]$ define e^k_n by $e^0_0(x) = 1$ and, when $n \ge 0$, $1 \le k \le 2^n$,

$$e^k_n(x) = \begin{cases} 2^{n/2} & \text{when } k - 1 \le 2^n x < k - 0.5, \\ -2^{n/2} & \text{when } k - 0.5 \le 2^n x < k, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the e^k_n form a unit–perpendicular basis for $L^2[0, 1]$. Hint: One route to showing that they span is first to show that if f is perpendicular to them all, then $\int_0^x f = 0$ for all x of the form $k2^{-n}$, and deduce that $\int_{\mathcal{B}} f = 0$ for every measurable $\mathcal{B} \subset [0, 1]$.

It is easily verified that $\|e^k_n\| = 1$ and $(e^j_m, e^k_n) = 0$ when $j, m \ne k, n$. Thus it suffices to show they span. To prove the first part of the hint, observe that if $x = k2^{-n}$, $I = [0, x)$ and c_I is the characteristic function of I, then $\int_0^x f = (f, c_I)$ and it would follow that this is 0 if c_I is a linear combination of the e^k_n. This is an easy proof by induction on n. Obviously if $x = 1$ we have $c_I = e^0_0$ and if $x = 0$, then $c_I = 0$. Now suppose that $n \ge 1$ and $1 \le k \le 2^n$. If k is even, then $k2^{-n} = (k/2)2^{n-1}$ and we can appeal to the inductive hypothesis. If k is odd, let $k' = (k + 1)/2$, $I_+ = [0, k'2^{1-n})$, $I_- = [0, (k' - 1)2^{1-n})$. Then $c_I = \frac{1}{2}(c_{I_-} + c_{I_+} - 2^{(n-1)/2} e^{k'}_{n-1})$. Now we have $\int_J f = 0$ whenever J is a subinterval of $[0, 1]$ with endpoints of the form $k2^{-n}$, Moroever given any measurable subset $\mathcal{B}$ of $[0, 1]$ and any $\varepsilon > 0$ there is a countable cover of $\mathcal{B}$ by intervals of the form J such that their union U satisfies $\mathrm{meas}(\mathcal{B}\backslash U) < \varepsilon$. From above $\int_U f = 0$ and by Schwarz $|\int_{\mathcal{B}\backslash U} f| \le \sqrt{\varepsilon}\|f\|_2$. It follows that $f = 0$ for almost all $x \in [0, 1]$

4. §1.3. Exercise 14. Show that the family $\{f_n\}$ spans $L^2(Q)$ iff $(f, f_n) = 0$ for every n implies $f \equiv 0$. Hint: What is the annihilator of the family?

(i) Suppose the f_n span $L^2(Q)$. Let f be such that $(f, f_n) = 0$ for every n. Choose c_n so that $\|f - \sum c_n f_n\| = 0$. Then squaring out $0 = \|f\|^2 - 2\Re\overline{c}_n(f, f_n) + \|\sum c_n f_n\|^2 = \|f\|^2 + \|\sum c_n f_n\|^2$. Hence $\|f\|^2 = 0$. (ii) Suppose that the f_n do not span $L^2(Q)$. We can certainly suppose that the f_n are independent, perpendicular and unimodular since we can still carry out the Gram–Schmidt diagonalization process and everything which is close to the original family will be close to the pruned one. Since the family does not span there will be an f such that $\|f - \sum_n d_n f_n\| > 0$ for every d_n. Let $d_n = (f, f_n)$ and $g = f - \sum d_n f_n$. Then $(g, f_n) = (f, f_n) - \sum_m d_m(f_m, f_n) = d_n - d_n\|f_n\|^2 = 0$ but $\|g\| > 0$.