# MATH 465 NUMBER THEORY, SPRING 2009, PROBLEMS 14

*Return by Monday 27th April*

1. Suppose that $p \equiv 1$ or $3 \pmod 8$.
   (i) Prove that there is a $u \in \mathbb{Z}$ such that $u^2 \equiv -2 \pmod p$.
   (ii) Deduce that there are $x \in \mathbb{Z}$, $y \in \mathbb{Z}$, not both 0, such that $|x| < \sqrt{p}$, $|y| < \sqrt{p}$ and $x^2 + 2y^2 \equiv 0 \pmod p$.
   (iii) Deduce that there are $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that either $x^2 + 2y^2 = p$ or $x^2 + 2y^2 = 2p$.
   (iv) Prove that if $x^2 + 2y^2 = 2p$, then there are $x_1 \in \mathbb{Z}$ and $y_1 \in \mathbb{Z}$ such that $2x_1^2 + y_1^2 = p$.

2. Find $d(300)$ and $\sigma(300)$.

3. Prove that, for every $n \in \mathbb{N}$ we have

$$\sum_{m|n} \mu(m)\sigma(n/m) = n.$$

4. For each $n \in \mathbb{N}$, let $f(n) = \sum_{m|n} d(m)^3$ and $g(n) = \sum_{m|n} d(m)$.
   (i) Prove that $f \in \mathcal{M}$ and $g \in \mathcal{M}$.
   (ii) Prove that

$$\sum_{j=0}^{k} (j+1) = \frac{1}{2}(k+1)(k+2).$$

   (iii) Prove that

$$\sum_{j=0}^{k} (j+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2.$$

   (iv) Prove that, for every $n \in \mathbb{N}$, $f(n) = g(n)^2$.