MATH 465 NUMBER THEORY, SPRING 2009, PROBLEMS 11

Return by Monday 6th April

Complex numbers and the exponential function

This homework is not connected with the more recent classwork, but does use some theory of divisiblity and residue classes. Let i denote the complex number $\sqrt{-1}$. Just about the only things you need to know about complex numbers are

A. For any real number α, $e^{i\alpha} = \cos\alpha + i\sin\alpha$ (Demoivre's theorem),

B. For any two complex numbers z and w we have $e^{z+w} = e^z e^w$.

For each real number α define $e(\alpha) = e^{2\pi i\alpha}$.

1. Prove that $e(\alpha)$ is periodic with period 1.

2. Prove that $e(\alpha) = 1$ if and only if $\alpha \in \mathbb{Z}$.

3. Let $m \in \mathbb{N}$, $a, b \in \mathbb{Z}$. Prove that $a \equiv b \pmod{m}$ if and only if $e(a/m) = e(b/m)$.

4. Prove that
$$\sum_{k=0}^{m-1} e(ak/m) = \begin{cases} m & \text{when } m|a, \\ 0 & \text{when } m \nmid a. \end{cases}$$

5. Let
$$c_m(a) = \sum_{\substack{k=1 \\ (k,m)=1}}^{m} e(ak/m),$$
i.e. a sum in which the index k is restricted to a set of reduced residues modulo m. This is Ramanujan's sum. Prove that if $p \nmid a$, then $c_p(a) = -1$ and that if $p|a$, then $c_p(a) = p - 1$.

6. Show that if $(m_1, m_2) = 1$, then $c_{m_1 m_2}(a) = c_{m_1}(a)c_{m_2}(a)$. Hint: Use the fact that $k_1 m_2 + k_2 m_1$ ranges over a reduced set of residues modulo $m_1 m_2$ as k_1 and k_2 range over a reduced set of residues modulo m_1 and m_2 respectively.