MATH 465 NUMBER THEORY, SPRING 2009, PROBLEMS 8

Return by Monday 16th March

1. (i) Let $m \in \mathbb{N}$. Prove that

$$(y-1)(y^{m-1}+y^{m-2}+\cdots+y+1)=y^m-1.$$

(ii) Let $n \in \mathbb{N}$. Prove that

$$(x^2+1)(x^2-1)(x^{4n-4}+x^{4n-8}+\cdots+x^4+1)=x^{4n}-1.$$

(iii) Let p be a prime number with $p \equiv 1 \pmod 4$. Prove that $x^2 \equiv -1 \pmod p$ has exactly two solutions.

2. Let $a \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\rho(n;a)$ denote the number of solutions of $x^2 \equiv a \pmod n$.

(i) Prove that if p is an odd prime and $a \not\equiv 0 \pmod p$, then $\rho(p;a) = 2$ or 0 (it is helpful to prove that if x_0 is one solution, then $-x_0$ is in a different residue class modulo p).

(ii) Prove that $\rho(n;a)$ is a multiplicative function of n, i.e. that if $(n_1,n_2)=1$, then $\rho(n_1n_2;a)=\rho(n_1;a)\rho(n_2;a)$.

(iii) Suppose that $p_1,\dots,p_k$ are different odd primes and $(a,p_1\dots p_k)=1$. Prove that $\rho(p_1\dots p_k;a)=2^k$ or 0.

3. (i) Solve $x^2+x+47 \equiv 0 \pmod 7$.

(ii) Use the Hensel-Newton method to find all solutions to

$$x^2+x+47 \equiv 0 \pmod{7^2}$$

and

$$x^2+x+47 \equiv 0 \pmod{7^3}.$$