MATH 465 NUMBER THEORY, SPRING TERM 2009, PRACTISE FINAL

Note: The Final Exam for Math 465 will be Monday 4th May, 2:30pm–4:20pm in room 116 Osmond.

1. Find $(1745, 1485)$ and integers x and y such that $1745x + 1485y = (1745, 1485)$.

2. Let x and y be integers which are not both 0. Suppose that $d \in \mathbb{N}$ is such that $d|x$, $d|y$ and, whenever $m \in \mathbb{N}$ and $m|x$ and $m|y$, one has $m|d$.
   (i) Prove that d is unique. This is the greatest common divisor (x, y).
   (ii) Prove that if u and v are integers which are not both 0, then $(u, v) = (u, u+v)$.

3. Solve the simultaneous congruences $x \equiv 3 \pmod 4$, $x \equiv 2 \pmod 7$, $x \equiv 7 \pmod 9$.

4. Find all solutions to the congruence $9x^{58} + 4x^{30} + 2x \equiv 0 \pmod{29}$.

5. (i) Solve $f(x) = x^3 - x - 1 \equiv 0 \pmod 5$.
   (ii) Use the Hensel-Newton method to find all solutions to

$$f(x) \equiv 0 \pmod{5^2}.$$

6. Show that 3 is a primitive root modulo 17 and draw up a table of indices to this base. Hence, or otherwise, find all solutions to the following congruences.
   (i) $x^{16} \equiv 3 \pmod{17}$,
   (ii) $x^{21} \equiv 3 \pmod{17}$,
   (iii) $x^{30} \equiv 8 \pmod{17}$.

7. Evaluate the following Legendre symbols, showing your working.

$$\text{(i)} \quad \left(\frac{-1}{103}\right)_L, \quad \text{(ii)} \quad \left(\frac{2}{103}\right)_L, \quad \text{(iii)} \quad \left(\frac{7}{103}\right)_L, \quad \text{(iv)} \quad \left(\frac{83}{103}\right)_L.$$

8. Find all solutions to the diophantine equation $x^2 + y^2 = 3z^2 + 3t^2$.