MATH 421 COMPLEX ANALYSIS, FALL TERM 2004, PRACTICE EXAM 1 SOLUTIONS

The first exam is on Wednesday 6th October, at 9:05 in 100 Boucke.
1. (i) Find the absolute value of $\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$. $\left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right| = \frac{|3+4i||-1+2i|}{|-1-i||3-i|} = \sqrt{\frac{(3^2+4^2)(1^2+2^2)}{(1^2+1^2)(3^2+1^2)}} = \sqrt{\frac{25.5}{2.10}} = \frac{5}{2}$. (ii) Show that if $a \neq 0$, then $\frac{1}{a} = \frac{\bar{a}}{|a|^2}$, and find the real part of $\frac{4-3i}{-1+i}$. Two solutions to first part. (1) Let $a = u+iv$ where u and v are real. Then $a^{-1} = u/(u^2+v^2) - iv/(u^2+v^2) = (u-iv)/(u^2+v^2) = \bar{a}/|a|^2$. (2) We have $|a|^2 = a\bar{a}$, and $|a| = 0$ iff $a = 0$. Thus $|a| \neq 0$. Hence we can divide both sides of this by $a|a|^2$. $\Re\left(\frac{4-3i}{-1+i}\right) = \Re\left(\frac{(4-3i)(-1-i)}{|-1+i|^2}\right) = \Re\left(\frac{-7-i}{2}\right) = -\frac{7}{2}$.

2. Find the image under the Möbius transformation $z \mapsto w : w = \frac{z-1}{z+1}$, of (a) the circle $|z+2| = 1$, (b) the line $\Re z = \Im z$. The inverse is given by $z = \frac{-w-1}{w-1}$. (a) This is the set of points w such that $1 = \left|\frac{w-3}{w-1}\right|$ and so is a stgright line, through the points $w = 2$ and $w = 2+i$. (b) Another way of writing the line is as the set of z such that $|z+1-i| = |z-1+i|$. Substituting for w gives the set of w such that $\left|\frac{w-1-2i}{w+i(2+i)/5}\right| = \sqrt{5}$.

3. Sketch the set of points z determined by the given condition. (a) $|z-1-i| \neq |z+1+i|$, (b) $|z-i-2| > 3$. Which, if any, of these sets are regions? (a) Two half planes separated by the straight line through 0 and $-1+i$. A union of two disjoint subsets, so not connected, so not a region. (b) Open annulus centred at $2+i$ and inner radius 3, extending to ∞. Polynomially connected, so connected.

4. (a) Prove that $\{z : 0 < \Re z < 1\}$ is an open set in $\mathbb{C}$. (b) Prove that if S and T are closed sets in $\mathbb{C}$, then so is $S \cup T$. (a) Let $S = \{z : 0 < \Re z \leq 0\}$ and let $z \in S$. Let $\delta = \min\{\Re z, 1 - \Re z\}$. Now let $w \in D(z,\delta)$. Then $|\Re w - \Re z| \leq |w - z| < \delta$, and so $\Re w = \Re z + (\Re w - \Re z) > \Re z - \delta \geq \delta - \delta = 0$ and $\Re w = \Re z + (\Re w - \Re z) < \Re z + \delta \leq 1 - \delta + \delta = 1$. Hence $w \in S$ and so $D(z,\delta) \subseteq S$. (b) Let $U = S \cup T$ and use * to denote the complement with respect to $\mathbb{C}$. Since S and T are closed, S^* and T^* are open. Let $z \in U^*$. Then $z \in S^*$ and $z \in T^*$, and so there are $r_1 > 0$, $r_2 > 0$ such that $D(z,r_1) \subseteq S^*$ and $D(z,r_2) \subseteq T^*$. Let $r = \min\{r_1, r_2\}$. Then $D(z,r) \subseteq S^*$ and $D(z,r) \subseteq T^*$ and so $D(z,r) \subseteq S^* \cap T^* = U^*$. Hence U^* is open and thus U is closed.