MATH 401 INTRODUCTION TO ANALYSIS-I, FALL TERM 2009, PROBLEMS 10

Return by Monday 2nd November

1. Prove, using only definitions and results established in the course, that

$$\text{(i)} \quad \lim_{n\to\infty} \left\{ \frac{4n^5 + 5n^3 + 6n}{2n^5 + 1} \right\} = 2,$$

$$\text{(ii)} \quad \lim_{n\to\infty} \frac{(-1)^n n}{n^2 + 1} = 0,$$

2. Suppose that $0 < k < 1$ and $\langle x_n \rangle$ satisfies $|x_{n+1}| < k|x_n|$ for $n = 1, 2, 3, \ldots$. Prove that

(i) $|x_n| \le k^{n-1}|x_1|$,

(ii) $\lim_{n\to\infty} x_n = 0$.

3. Prove that if $x > 0$ and $\langle x_n \rangle$ is a sequence with $\lim_{n\to\infty} x_n = x$, then there is a real number N such that whenever $n > N$ we have $x_n > 0$.

4. (i) Prove that, if $n \in \mathbb{N}$ and $n \ge 4$, then $2^n < n!$, and deduce that if $n \ge 5$, then $2^n \le 2((n-1)!)$.

(ii) Prove that

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$