MATH 401, FALL TERM 2009, MODEL SOLUTIONS TO PRACTICE EXAM 2

Note that Exam 2 is on Monday 26th October in Room 207 Sackett, 1:25-2:15

1. Suppose that x is a real number with $x > 1$. (i) Prove that $x < x^3$. (ii) Prove that $1 < x^5 < x^7$.

(i) We have $x < 1$, i.e. $1 < x$. Also $0 < 1$. Thus, by (O2), $0 < x$. Thus, by (O4), $x = 1.x < x.x = x^2$ and $x^2 = 1.x^2 < x.x^2 = x^3$. Thus, by (O2), $x < x^3$. (ii) $x > 0$ so by (O4), $x^2 = x.x < x^3.x = x^4$, $x^3 = x.x^2 < x^3.x^2 = x^5$, $x^5 = 1.x^5 < x.x^5 = x^6$, $x^6 = x.x^5 < x.x^6 = x^7$. Hence, by repeated application of (O2), $1 < x < x^3 < x^4 < x^5$ and $x^5 < x^6 < x^7$.

2. Determine the set $\mathcal{A} = \left\{x : \frac{x+5}{x^2+2} < \frac{2}{x}\right\}$.

We have $x^2 + 2 \geq 2 > 0$ always. (a) First consider $x > 0$. Then $\frac{x+5}{x^2+2} < \frac{2}{x}$ iff $x(x+5) < 2(x^2+2)$ iff $x^2 + 5x < 2x^2 + 4$ iff $0 < x^2 - 5x + 4$ iff $0 < (x-4)(x-1)$. This holds iff either $x-4 > 0$ and $x-1 > 0$ OR $x-1 < 0$ and $x-4 < 0$. Thus $x > 4$ or $0 < x < 1$. (b) Now suppose $x < 0$. Then $\frac{x+5}{x^2+2} < \frac{2}{x}$ holds iff $0 > (x-4)(x-1)$. This holds iff *either* $x - 4 > 0$ and $x - 1 < 0$ *or* $x - 1 > 0$ and $x - 4 < 0$. In either case $x > 1$ contradicting $x < 0$. Thus the complete answer is $\mathcal{A} = (0, 1) \cup (4, +\infty)$.

3. Prove that there is no rational number x such that $x^2 = 5$.

Suppose that there is a rational solution. Thus we may suppose that $x = a/b$ with $a \in \mathbb{N}$ and $b =\in \mathbb{N}$, and a/b is in lowest terms. In particular one of a or b will NOT be a multiple of 5. Now $a^2 = 5b^2$. Thus a^2 is a multiple of 5. Hence a is a multiple of 5, so there is a $c \in \mathbb{N}$ such that $a = 5c$. Thus $5^2c^2 = 5b^2$ and so $5c^2 = b^2$. Thus b^2 is a multiple of 5, and therefore both a and b are multiples of 5 contradicting the fact that a/b is in lowest terms.

4. Prove that if $n \in \mathbb{N}$, then $3^n > n^2$.

We use induction on n. Case $n = 1$: We have $3^1 = 3 > 1 = 1^2$. Case $n = 2$: We have $3^2 = 9 > 4 = 2^2$. Inductive step: Suppose that $3^n > n^2$ and $n \geq 2$. Then $3^{n+1} = 3.3^n > 3n^2 = n^2 + 2n^2 \geq n^2 + 4n = n^2 + 2n + 2n > n^2 + 2n + 1 = (n+1)^2$.