Topological Implications of Amenability

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Make Money Fast!

Let X be a metric space. A *Ponzi scheme* on X is a locally finite 1-chain $c = \sum a_i[x_i, y_i]$ on X, where the real numbers a_i are uniformly bounded in absolute value and the 1-simplices $[x_i, y_i]$ are uniformly bounded in length.

A Ponzi scheme *succeeds* if the boundary $bc = \sum a_i([y_i] - [x_i])$ is uniformly locally positive (say, has mass greater than 1 on each ball of some radius r).

A metric space is *amenable* if it has no successful Ponzi scheme.

Examples: Every compact space is amenable. $\mathbb{R}^n$ is amenable. An infinite tree is *not* amenable.

Homework: Hyperbolic space is not amenable.

Amenable Groups

A finitely generated group Γ can be considered as a metric space via the *word length metric*. This is the largest metric on Γ such that $d(x, sx) \leq 1$ for each generator s. The metric depends on the generating set, but its 'large scale type' does not.

Theorem The following are equivalent:

- Γ is amenable as a metric space
- There is a Følner sequence in Γ
- There is an *invariant mean* $\ell^\infty(\Gamma) \to \mathbb{C}$
- The trivial representation is weakly contained in $\ell^2(\Gamma)$

Ahlfors and the Picard theorem

In *Acta* 1935, Ahlfors proved

Proposition Let $f\colon \mathbb{C} \to \mathbb{P}^1$ be a non-constant meromorphic function. Then $\mathbb{C}$, equipped with the metric pulled back via f, is amenable.

He used this to give a topological proof of Picard's theorem — a nonconstant meromorphic function cannot omit three values — and other results of Nevanlinna theory.

Basic idea: let $\Omega_1, \Omega_2, \Omega_3$ be small discs around the omitted points. The Riemann-Hurwitz relation predicts that the Euler characteristic of $f^{-1}(\mathbb{P}^1 \setminus \bigcup \Omega_j)$ should be highly negative. Amenability is used to show that we can't import enough positive Euler characteristic from infinity to circumvent the Riemann-Hurwitz relation.

Quasi-isometry to positive scalar curvature

Recall that a compact spin manifold M with nonzero $\widehat{\mathcal{A}}$-genus cannot carry positive scalar curvature.

In my thesis I asked: Can the metric on $\widetilde{M}$ be (unequivariantly) deformed within its quasi-isometry class to one of positive scalar curvature?

Theorem Such a deformation is possible if and only if $\pi_1(M)$ is not an amenable group.

The 'only if' direction (JR 1988) uses an index with values in the K-theory of an algebra of finite propagation operators, together with a trace on that algebra provided by amenability.

The 'if' direction (Block-Weinberger 1992) uses controlled surgery.

The Novikov Conjecture

The *structure set* $\mathcal{S}(X)$ of a Poincaré space X classifies manifold structures within the homotopy type of X. Surgery theory analyzes $\mathcal{S}(X)$ in terms of homotopy theory and quadratic forms. The main outstanding question is:

Borel Conjecture Manifold structures are classified by relative homology:

$$\mathcal{S}_\circ(X) = H_\circ(X \to B\pi_1(X); \mathbb{L}(e)).$$

One implication is the *Novikov Conjecture*: the higher signatures of a manifold M,

$$\langle \mathcal{L}(TM) \smile f^*(c), [M] \rangle, \qquad f\colon M \to B\pi,$$

are invariants of homotopy type.

Theorem: (Higson-Yu-JR 1998) If Γ acts amenably on any compact Hausdorff space, then the Novikov conjecture is true for fundamental group Γ.

What is an amenable action?

A discrete group Γ *acts amenably* on a compact space X if there is a sequence of (weak-$*$ continuous) maps $X \to \text{prob}(\Gamma)$ which is asymptotically equivariant.

For example, Γ acts amenably on a point if and only if it is in fact an amenable group.

But there are many other examples, for instance the action of a free group (or a hyperbolic group) on its natural boundary.

Remark: Amenability is functorial. Therefore, if Γ acts amenably on any compact Hausdorff space at all, then it acts amenably on its Stone-Čech compactification $\beta\Gamma$.

Outline of a proof

- If Γ acts amenably on $\beta\Gamma$, then Γ is coarsely equivalent to a subset of Hilbert space (one says: Γ embeds uniformly in Hilbert space.) **Homework:** ℓ^1 embeds uniformly in Hilbert space.

- If a metric space X embeds uniformly in Hilbert space then the Coarse Baum–Connes Conjecture (CBC) is true for X.

- The Coarse Baum–Connes conjecture for the underlying metric space of a group Γ implies the Novikov Conjecture for Γ.

CBC states that a certain assembly mapp

$$\mu: KX_*(X) \to K_*(C^*(X))$$

is an isomorphism. In other words, the K-theory of $C^*(X)$ can be computed entirely in terms of the indices of elliptic operators.

CBC Counterexamples

We look in the non-amenable world.

A fg group Γ has *Kazhdan's property T* if the trivial representation is *isolated* in the unitary dual. This is a strong form of non-amenability.

Let Γ be a property T group acting ergodically on S^{n-1} and form a metric space X from $\mathbb{R}^n$ by introducing 'shortcuts' along the Γ-action: take the largest metric that is dominated by the regular Euclidean metric and is such that $d(x, sx) \leq 1$ for each generator s. Then X (in fact, a closely related 'double' of X) is a counterexample to CBC. (Higson-Lafforgue-Skandalis 2000; JR).

Gromov asserts that one can construct *groups* containing this sort of geometry. They do not embed in Hilbert space, and are *potential counterexamples* to the Novikov conjecture.