Coarse Geometry and Contorted Cones

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Coarse Geometry

Let X and Y be metric spaces.

A *coarse map* $f\colon X \to Y$ is metrically proper (f^{-1} of a bounded set is bounded) and satisfies

$$\forall R > 0\, \exists S > 0\, d_X(x, x') < R \Rightarrow d_Y(f(x), f(x')) < S.$$

The spaces X and Y are *coarsely equivalent* if there exist coarse maps $X \to Y$, $Y \to X$ which are inverses up to uniformly bounded error. If X is coarsely equivalent to a subspace of Y we say that X is *uniformly embeddable* in Y.

Examples $\mathbb{R}$ is coarsely equivalent to $\mathbb{Z}$. The tree $\mathbb{F}_2$ is uniformly embeddable in Hilbert space.

The subsets S of $X \times X$ on which the distance function is bounded are called *controlled*. Can axiomatize the properties of controlled sets, but we'll consider only metric examples. *Coarse geometry* studies the properties of coarse spaces and maps.

Examples for Coarse Geometry

Compare Mostow, 1973 (‘Strong rigidity of locally symmetric spaces’). Three kinds of examples have been studied extensively:

- *Finitely generated groups* with word metrics.
- *Symmetric spaces* of non-compact type, and more general spaces of non-positive curvature.
- *Open cones* on compact spaces.

Of course there are connections between these various kinds of examples. For instance a uniform lattice in a Lie group is coarsely equivalent to the associated symmetric space; and the study of boundaries of nonpositively curved spaces can be thought of as relating these spaces to suitable open cones — as one sees most perfectly in the case of $\mathbb{R}^n$.

The purpose of this talk is to propose a new category of examples.

Coarse Geometry and Groupoids

Digression: Skandalis, Tu and Yu have succeeded in relating coarse geometry to the theory of groupoids. They define a functor

$$\text{Coarse geometry} \Longrightarrow \text{Topological groupoids.}$$

Let X be a coarse space, uniformly discrete for simplicity. Construct a groupoid $\mathcal{G}$ as follows: the objects are points $\omega \in \beta X$, the Stone-Cech compactification of X, and the morphisms $\omega \to \omega'$ are equivalence classes of *partial translations*.

Definition A map $\phi\colon X \to X$ is a *translation* if $d(x, \phi(x))$ is uniformly bounded.

Theorem (CSY) For $X = |\Gamma|$, $\mathcal{G} = \beta\Gamma \rtimes \Gamma$.

We will construct a functor in the opposite direction

$$\text{Group actions} \Longrightarrow \text{Coarse geometry.}$$

Contorted Cones

Situation: M a smooth compact manifold, Γ a finitely generated group of diffeomorphisms.

Give Γ a right-invariant word metric. Thus the right action of Γ on itself is by isometries, the left action is by translations.

The open cone $\mathcal{O}M$ is a proper metric space. The action of Γ on $\mathcal{O}M$ is not an action by translations. Let us contort the metric to make it so. Thus, give $\mathcal{O}M$ the greatest metric δ which obeys the inequalities $\delta(x, x') \leq d(x, x')$, the original metric on the open cone, and

$$\delta(x, \gamma x) \leq |\gamma|$$

for all $\gamma \in \Gamma$, where $|\gamma|$ denotes the word-length.

Theorem: This defines a proper metric δ on $\mathcal{O}X$. Up to coarse equivalence the metric is independent of the choices involved. Moreover, the contorted cone $\mathcal{O}_\Gamma(X)$ really does 'open out' (the diameter of the cross-sections tends to infinity) except in trivial cases.

Embedding Contorted Cones in H

Coarse spaces that uniformly embed in Hilbert space are the simplest.

Theorem If Γ is an amenable group, or more generally if Γ acts amenably on M, then $\mathcal{O}_\Gamma(M)$ uniformly embeds in a Hilbert space.

Proof proceeds by way of the 'property A' of Guoliang Yu, which implies uniform embeddability. A (uniformly discrete) space X has *property A* if there is a sequence of maps $a^n \colon X \to \mathrm{prob}(X)$ such that

- for every n there is $r > 0$ such that each a^n_x is supported in a r-ball around x;
- for every $r > 0$, $a^n_x - a^n_{x'}$ tends to zero in ℓ^1 norm, uniformly on $d(x, x') < r$.

Property A implies uniform embeddability in Hilbert space. (use negative type functions.)

Finite Propagation Operators

By an X-*module* we mean a Hilbert space H on which $C_0(X)$ acts. For instance, $H = L^2(X)$, or X is the spectrum of some normal operator on H.

Given such an H we can define the *support* of an operator $T \in \mathfrak{B}(H)$, as a subset of $X \times X$. The definition is so arranged that if $H = L^2(X, \mu)$ and T is an integral operator

$$Tu(x) = \int k(x, y) u(y)\, d\mu(y),$$

then the support of T is equal to the support (in the usual sense) of the function k.

Definition The operator T has *finite propagation* if its support is controlled.

The finite propagation operators on H form a $*$-algebra, $\mathcal{F}(X)$, which reflects the coarse structure of X.

Finite Propagation C^*-Algebras

Let $C^*(X)$ be the C^*-algebra (norm closed $*$-algebra of operators on Hilbert space) generated by the *locally compact* finite propagation operators. This algebra was introduced to study index theory on open manifolds.

Example Let $X = \mathbb{Z}$ and let $H = L^2(S^1)$, made into an X-module via the spectral theory of the Dirac operator $id/d\theta$. The $C(S^1) \subseteq C^*(X)$.

The *coarse Baum–Connes conjecture* proposes a calculation for the K-theory groups $K_i(C^*(X))$. This conjecture is true if X is uniformly embeddable into Hilbert space.

For our purposes we need only know that CBC implies that every class say in $K_0(C^*(X))$ is representable by an idempotent which is genuinely of finite propagation (and not merely a norm limit of finite propagation operators.)

The transfer homomorphism

Will show that certain contorted cones *fail* to satisfy the coarse Baum–Connes conjecture. Let G be a compact Lie group with a bi-invariant metric, and Γ is a finitely generated subgroup acting by translations.

Fix $r > 0$. Far to the right in the contorted cone $\mathcal{O}_\Gamma(G)$, the contorted r-neighborhood of the diagonal splits up into components parametrized by $\gamma \in \Gamma$ with $|\gamma| \leq r$; it is contained in the *disjoint* union

$$\bigsqcup_{|\gamma| \leq r} \{(x, \gamma x') : d(x, x') \leq r\}$$

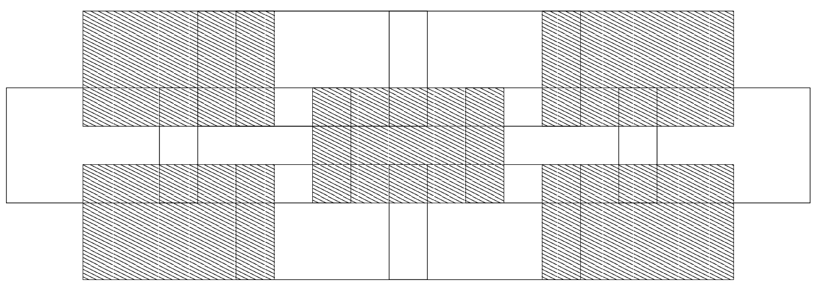
where d denotes the ordinary (uncontorted) distance. This gives a *transfer* $*$-homomorphism

$$FPLC(\mathcal{O}_\Gamma(G)) \to FPLC(\mathcal{O}(G); C_r^*(\Gamma)).$$

(For the STY-groupoids this corresponds to the assertion that, at infinity, $\mathcal{G}(\mathcal{O}_\Gamma(G))$ is isomorphic to $\mathcal{G}(\mathcal{O}(G)) \rtimes \Gamma$).) **Question:** Does this $*$-homomorphism extend continuously to the generated C^*-algebras?

Finite asymptotic dimension

Recall that a (compact) metric space has topological dimension $\leq d$ if for each $\varepsilon > 0$ it has an open cover by at most $d+1$ 'blocks', each block being a union of disjoint open pieces each of which has diameter $< \varepsilon$.



We say that X has *asymptotic dimension* $\leq d$ if for each $r > 0$ it has a cover by at most $d+1$ 'r-separated blocks'; such a block is a disjoint union of pieces of uniformly bounded diameter, and the pieces must be at least r apart.

Theorem If $|\Gamma|$ has finite asymptotic dimension, then the transfer homomorphism extends continuously to the C^*-algebras.

Property T

Recall the mean ergodic theorem: let $\rho\colon \Gamma \to \mathfrak{B}(H)$ be a unitary representation of the group $\Gamma = \mathbb{Z}$. Then the sequence of elements $p_n(z) = (1 + \cdots + z^{n-1})/n$ in the group ring $\mathbb{C}[\Gamma]$ has the property that $\rho(p_n)$ converges $*$-strongly to the projection P onto the Γ-invariant subspace of H.

For some groups Γ (those with Kazhdan's property T) one can find a sequence p_n in $\mathbb{C}[\Gamma]$ such that $\rho(p_n) \to P$ *in norm*. Otherwise stated, the maximal C^*-algebra $C^*_{\max}(\Gamma)$ contains a projection (the *Kazhdan projection*) which maps to the projection on the invariant elements in any unitary representation.

A Contorted Counterexample to CBC

Let $G = SO(n)$, $n \geq 5$, and let Γ be a discrete subgroup that is dense, hyperbolic and has property T.

Because Γ acts by translations on $\mathcal{O}_\Gamma(G)$, there is a $*$-homomorphism from $C^*_{\max}(\Gamma)$ to the multiplier algebra of $C^*(\mathcal{O}_\Gamma(G)$. Under this $*$-homomorphism, the Kazhdan projection is sent to the projection p onto *radial functions*.

Note that, contrary to appearances, p is a norm limit of finite propagation operators.

The external product of p with the Bott generator for K_1 of the finite propagation operators on the line gives a class in $K_1(C^*(\mathcal{O}_\Gamma(X)))$ which *does not* belong to the image of the assembly map.

Why not?

$$
\begin{array}{ccc}
K_1(C^*(\mathcal{O}_\Gamma(X))/\mathfrak{K}) & \xrightarrow{\text{transfer}} & K_1((C^*(\mathcal{O}(X);C_r^*(\Gamma))/\mathfrak{K})) \\
\Big\downarrow \text{crush} & & \Big\downarrow \text{crush} \\
K_1(C^*(|\mathbb{R}^+|)/\mathfrak{K}) & & K_1((C^*(|\mathbb{R}^+|;C_r^*(\Gamma))/\mathfrak{K}) \\
\Big\downarrow \text{connect} & & \Big\downarrow \text{connect} \\
K_0(\mathfrak{K}) & & K_0(C_r^*(\Gamma)) \\
\Big\downarrow = & & \Big\downarrow \text{trace} \\
\mathbb{Z} & \overset{\text{finite propagation}}{=\!=\!=\!=\!=\!=\!=} & \mathbb{R}
\end{array}
$$