The Novikov Conjecture in Low Dimensions: History and Agenda

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Novikov's paper

The history of the Novikov Conjecture begins with the Hirzebruch Signature Theorem: Let M^{4k} be a compact oriented manifold, then

$$\mathrm{Sign}(M) = \langle L_{4k}(M), [M] \rangle \tag{1}$$

As a consequence, $L_{4k}(M^{4k})$ is a topological invariant — even an invariant of (oriented) homotopy type.

QUESTION: Which other components of L are topologically invariant? (In other words, what characteristic classes exist for 'topological tangent bundles'.)

ANSWER: (eventually) All of them.

First major work was by Novikov 1965 in *Rational Pontrjagin classes, homeomorphism and homotopy type of closed manifolds* (Iszvestija 29(1965), 1373-1388): he proved among other things that $L_{4k}(M^{4k+1})$ is topologically invariant. (This had earlier been observed by Rochlin.)

Novikov's method

It is enough to show that $\langle L_{4k}(M^{4k+1}), [N]\rangle$ is topologically invariant, where $[N]$ is the homology class corresponding to a $4k$-dimensional cycle N. If N is a closed (oriented) submanifold, this quantity is actually the signature of N.

Equivalently, consider

$$\langle L_{4k}(M^{4k+1}) \smile x, [M]\rangle \qquad (2)$$

where $x \in H^1(M; \mathbb{Q})$ is Poincaré dual to N.

(3) LEMMA: *Every integral 1-cohomology class $x \in H^1(M; \mathbb{Z})$ is pulled back from the circle via a map $p\colon M \to S^1$.*

For proof, just remark that S^1 is an Eilenberg MacLane space $K(\mathbb{Z}, 1)$ — or, explicitly, try to define a map to $\mathbb{R}$ by integration, then mod out by the integer periods. Note that by Sard's theorem one can take $N = p^{-1}\{0\}$ a submanifold.

One now passes to an infinite cyclic cover $\widetilde{M}$ of M associated to p, and observes that the N and N' associated to two smooth structures can be taken to be topologically cobordant in $\widetilde{M}$.

Equivalently (and this is what Novikov actually does in the cited paper) one shows that the signature of N can be recovered from the algebraic properties of the (infinite-dimensional) cohomology ring $H^*(\widetilde{M};\mathbb{Q})$, together with the extra structure provided by the 'control map' p.

(4) REMARK: The relation between the preceding two paragraphs is the same as the relation between the cobordism invariance of the analytic index and the index theorem for partitioned manifolds (cf. Higson's proof of cobordism invariance.)

Nowadays one would say that this argument proves the homotopy invariance of the higher signature associated to the class x pulled back from $S^1 = B\mathbb{Z}$.

What was important was a specific geometric model for 1-dimensional cohomology classes, in terms of maps to the circle, and infinite cyclic covers.

2-dimensional classes

One might hope similarly that every 2-dimensional cohomology class is geometrically represented by pulling back via a map $p\colon M \to X$, where X is some finite-dimensional universal space (e.g. a surface, if one is very rash). However the cup product shows that such hopes are doomed to disappointment.

What is true is

(5) LEMMA: *Every integral 2-cohomology class $x \in H^2(M;\mathbb{Z})$ is pulled back from $\mathbb{C}P^\infty$ via a map $p\colon M \to \mathbb{C}P^\infty$. Equivalently, every such class is the first Chern class of a complex line bundle.*

For proof, consider sheaf cohomology.

Addendum to the lemma: Every 2-form in the de Rham cohomology class of x is ($i/2\pi$ times) the curvature of a connection on the representative line bundle.

Almost flat bundles

Connes-Gromov-Moscovici, *Conjecture de Novikov et fibrés presque plats*, C.R. Acad Sci Paris 310(1990), 273-277:

> Ce théorème démontre également en toute géneralité la conjecture de Novikov pour le sous-anneau de $H^*(B\Gamma)$ engendré par $H^n(B\Gamma)$, $n \leqslant 2$.

In truth, there is no such thing as a single 'almost flat bundle' (any more than there is such a thing as a single 'almost invariant vector', in the context of property T).

What CGM did was to define a K-theory class to be 'almost flat' if, for every $\varepsilon > 0$, it can be represented as a formal difference of complex vector bundles that admit Hermitian structures and compatible connections, with curvature less than ε in norm.

Although few K-theory classes are represented by flat bundles (the Chern character must vanish), many classes are almost flat in this sense.

(6) Remark: Typically, the fiber dimension of the representing bundles tends to infinity as ε tends to 0.

(7) Theorem: *If $x \in K^0(B\Gamma)$ is an almost flat K-theory class, then the higher signature (for manifolds with fundamental group Γ) corresponding to $\mathrm{ch}(x)$ is a homotopy invariant.*

This is due to CGM, *loc. cit.*; see also the paper of Hilsum-Skandalis, Crelle 423(1992), 73–99. In fact what is true is that if a K-theory class has a 'flat enough' representative, then the associated higher signature is invariant under some given homotopy equivalence. The 'enough' in 'flat enough' depends on the homotopy equivalence in question.

But, how does this apply to 2-dimensional classes?

2-dimensional argument

Imagine that B is a manifold and that $x \in H^2(B;\mathbb{Z})$. Suppose further that B has a sequence of finite coverings $\pi_n \colon B_n \to B$, of degree $m_n = [B : B_n]$, such that the pull-back of x to B_n is divisible by m_n.

(Think $B = B\Gamma$ where Γ is residually finite.)

Let L be a line bundle over B with Chern class x. Then $\pi_n^*(L)$ has an m_n'th root, L_n, whose curvature is of order m_n^{-1}.

One can now push forward (this is the transfer homomorphism in K-theory) to obtain a bundle $V_n = \pi_{n\,!} L_n$ over B, of rank m_n, whose curvature is still of order m_n^{-1}. Moreover,

$$\mathrm{ch}(V_n) = m_n \exp(x/m_n) \in H^{**}(B;\mathbb{Q}) \qquad (8)$$

Now suppose that we have a homotopy equivalence of manifolds over B, i.e. a commuting diagram

$$\begin{array}{ccc} M & \longrightarrow & M' \\ & {\scriptstyle p}\searrow & \downarrow {\scriptstyle p'} \\ & & B \end{array}$$

where the horizontal map is a homotopy equivalence. For all sufficiently large n the bundles V_n will be 'flat enough', relative to the given homotopy equivalence, for Theorem 7 to apply. Thus we will have

$$\langle L(M) \smile p^* \mathrm{ch}(V_n), [M] \rangle = \langle L(M') \smile p'^* \mathrm{ch}(V_n), [M'] \rangle \tag{9}$$

for all sufficiently large n.

Taking into account Equation 8, this shows that the expression

$$\sum_\ell \frac{\alpha^\ell}{\ell!} \langle L_{\mathrm{dim}(M)-2\ell}(M) p^*(x)^\ell, [M] \rangle \tag{10}$$

is equal to the corresponding primed expression for a sequence of values of $\alpha = m_n^{-1}$ tending to zero. The coefficients of these expressions, i.e. the higher signatures corresponding to x and its powers, must therefore agree.

Infinite dimensional almost flat bundles

All this argument depended on the 'residual finiteness type' assumption that B admits a good sequence of finite covers.

The idea for the general case is sketched by Gromov in *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, sections $9\frac{1}{8}$ and $9\frac{1}{7}$. Namely, one should now look at the universal cover $\widetilde{B}$ of $B = B\Gamma$. Pulling back L to $\tilde{B}$ necessarily yields a trivial bundle (since $\tilde{B}$ is contractible), which therefore has roots L_d of all orders. One should use an appropriate L^2-index theorem on the universal cover to prove the homotopy invariance with coefficients in these L_d. Equivalently, we are now looking at infinite dimensional almost flat bundles on B — the push-forwards of the L_d to B.

(11) REMARK: This progression from 'large finite covers' to 'infinite covers' parallels the Gromov-Lawson work on positive scalar curvature in the seventies and eighties.

Digression: Connes' embedding property

It turns out that almost flat theory has a connection to a famous question in von Neumann algebra theory.

Let R be the hyperfinite II_1 factor. Then R^ω denotes the ultraproduct of (countably many) copies of R; it is also a II_1 factor. Connes asked — is R^ω universal for (separable) II_1 factors; i.e., is every such factor, M, a subfactor of R^ω.

(12) REMARK: Ozawa recently proved that there is no separable universal II_1 factor.

One way to embed $\mathcal{N}(\Gamma)$ in R^ω is to make use of 'almost homomorphisms' from Γ to finite-dimensional unitary groups; such almost homomorphisms arise from the holonomy of almost flat bundles. (A group whose von Neumann algebra embeds in this way is sometimes called a hyper-linear group.)

Mathai's method: twisted convolution algebras

Recall Atiyah's L^2 index theorem: given an elliptic operator D on a compact manifold M, and a Galois covering $\widetilde{M}$ of M with Galois group Γ, one can consider the lifted elliptic operator $\widetilde{D}$ on $\widetilde{M}$. This is a Γ-invariant elliptic operator and so it has an index in $K_*(C_r^*(\Gamma))$.

(13) THEOREM: *We have*

$$\tau_*(\mathrm{Index}(\widetilde{D})) = \mathrm{Index}(D)$$

where τ is the canonical trace on $C_r^*(\Gamma)$.

Mathai — If we twist the lifted operator $\widetilde{D}$ by a fractional power of a line bundle pulled back from M, the resulting operator will have an index in the K-theory of a twisted convolution algebra.

(14) DEFINITION: *Let Γ be a group. A $\mathbb{T}$-valued 2-cocycle on Γ is a function $\sigma : \Gamma \times \Gamma \to \mathbb{T}$ which vanishes when one of its arguments is the identity (normalization condition) and satisfies the cocycle condition*

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3) \qquad (15)$$

Given such one can define the twisted group algebra $\mathbb{C}[\Gamma, \sigma]$ to be the algebra linearly spanned by the group elements, with multiplication determined by $[g_1] \cdot [g_2] = \sigma(g_1, g_2)[g_1 g_2]$. The cocycle condition (15) ensures that this multiplication is associative.

There is a natural 'regular' representation and completing with respect to this we obtain the twisted group C^*-algebra $C_r^*(\Gamma, \sigma)$.

Consider now the case of a line bundle L over $B = B\Gamma$, and equip it with some (arbitrarily chosen) connection. Its Chern class x is an integral cohomology class for G, $x \in H^2(G;\mathbb{Z})$.

For $\alpha > 0$ let L_α be the line bundle (with connection) on $\widetilde{B}$ whose curvature is α times the lift of L. One may regard this as a bundle with Chern class $\alpha x \in H^2(G;\mathbb{R})$. Exponentiating, one obtains a $\mathbb{T}$-valued group cocycle $\sigma \in H^2(G;\mathbb{T})$, via the Bockstein homomorphism associated to

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

Let $p\colon M \to B$ be a manifold over B.

(16) PROPOSITION: *The signature operator on $\widetilde{M}$, twisted by $p^* L_\alpha$, has an index in $K_0(C^*_r(\Gamma,\sigma))$. Moreover, the trace of this index is given by the formula*

$$\sum_\ell \frac{\alpha^\ell}{\ell!} \langle L_{\dim(M)-2\ell}(M) p^*(x)^\ell, [M] \rangle \qquad (17)$$

which we have already seen in our earlier discussion.

One shows also that a homotopy equivalence of manifolds over B preserves the L_α-twisted signature, provided that α is small enough. As before, ‘small enough’ is relative to the particular homotopy being considered.

From this one can deduce the homotopy invariance of the higher signatures associated to x and its powers, in the same way as in the earlier discussion.

Coarse geometry

Let X be a coarse space, $y \in HX^q(X;\mathbb{Z})$ a coarse cohomology class. One could say (following Gromov-Lawson) that y is hyper-euclidean if it is pulled back from the fundamental class in $HX^q(\mathbb{R}^q)$ via a coarse map $X \to \mathbb{R}^q$.

- Hyper-euclidean classes satisfy (some) coarse counterparts of the Novikov conjecture.

- Every 1-dimensional coarse class (for a coarsely geodesic space) is hyper-euclidean.

- In dimension 3 there exist non-hyper-euclidean classes (use the secondary product on coarse cohomology).

Are there reasonable conditions on X which imply that all classes in $HX^2(X)$ are hyper-euclidean?

Concluding thoughts

- The key idea of all known proofs of Novikov for 2D classes is the same — twist by small powers of the representative line bundle.

- Is there a purely topological proof (cf Atiyah-Hirzebruch for manifolds with S^1 actions)?

- Higher dimensions,e.g. model $K(\mathbb{Z}, 3) = BPU(\infty)$?