Coarse Index Theory
Lecture 5

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Outline

Definition

Let X be a discrete bg coarse space and let E be a controlled set. The Rips complex $P_E(X)$ is the simplicial complex whose vertices are the points of X, and such that $x_0, \ldots, x_p$ span a p-simplex iff $\{x_0, \ldots, x_p\}$ is contained in some E-ball.

If $E \subseteq E'$ then $P_E(X)$ is a subcomplex of $P_{E'}(X)$. Therefore, the homology groups $K_*(P_E(X))$ form a direct system.

Definition

The coarse K-homology of X, denoted $KX_*(X)$, is the direct limit

$$\lim_{\substack{\rightarrow \\ E}} K_*(P_E(X)).$$

Passing to the limit we obtain the coarse assembly map

$$KX_*(X) \to K_*(C^*(X))$$

for any (discrete, but this condition can be removed) coarse space X.
It is natural to conjecture

Conjecture

The coarse assembly map is always an isomorphism (?)

The work of Guoliang Yu has shown that this conjecture is true in very many cases (e.g., whenever X can be coarsely embedded in a Hilbert space). However, it is false in general! (Higson 2000)

Box spaces

Let G be a finitely generated discrete group.

Definition

G is residually finite if there exists a sequence G_n of finite index normal subgroups of G with $\bigcap G_n = \{1\}$.

If G is residually finite, we define the box space $\Box G$ to be the coarse disjoint union $\bigsqcup G/G_n$. Notice that G acts on each G/G_n and thus on $\Box G$; we define the coarse structure on $\Box G$ by requiring that E is controlled if and only if there is a finite subset F of G such that, for each $(x, y) \in E$, there is $g \in F$ such that $x = gy$.

Definition

A translation of a coarse space X is a bijection $X \to X$ whose graph is controlled.

Proposition

Let G be a (discrete) group that acts on a (discrete) coarse space X by translations. There is an induced $*$-homomorphism

$$C^*_{\max}(G) \to C^*(X).$$

It is important that we use the maximal C^*-algebra here. In particular, we get a homomorphism $C^*_{\max}(G) \to C^*(\Box G)$.

Property T

Definition

Let G be a discrete group. G has property T if there exists a projection $p \in C^*_{\max}(G)$ that has the following property: for every representation $\rho\colon G \to U(H)$, the image $\rho(p) \in \mathfrak{B}(H)$ is the orthogonal projection onto the subspace of G-invariant vectors.

The definition is due to Kazhdan, and p is called the Kazhdan projection. Obviously every finite group has property T, but there exist infinite examples also (this is rather surprising).

Let G be an infinite, residually finite, property T group (e.g. $G = SL(3, \mathbb{Z})$) and let $X = \Box G$. Let $q \in C^*(X)$ be the image of the Kazhdan projection p under the canonical homomorphism $C^*_{\max}(G) \to C^*(X)$.

We know that q is the orthogonal projection onto the G-invariant functions, i.e. those functions constant on each coarse component of X. In particular, q is a projection with infinite-dimensional range.

Remark

q is not a controlled operator. Property T tells us that it is a limit of controlled operators, but this is mysterious!

Higson showed that (under appropriate hypotheses, fulfilled e.g. in the case of $SL(3, \mathbb{Z})$) the K-theory class $[q] \in K_0(C^*(X))$ does not belong to the image of the coarse assembly map. We'll prove a related but weaker result, namely that q is not a limit of finite propagation idempotents. (The relationship is that any index can, as we know, be represented by an idempotent of finite propagation.)

Definition

(Yu) Let X be a discrete coarse space. An operator $T \in C^*(X)$ is a ghost if its matrix entries tend to 0.

The ghosts form an ideal in $C^*(X)$ which is (possibly) not 'geometric', i.e. not of the form I_Y.

Lemma

Every compact operator is a ghost, and every finite propagation ghost is compact.

The element $q \in C^*(\Box G)$ coming from the Kazhdan projection is a ghost but not compact (it is a projection with infinite-dimensional range).
Suppose that there is a sequence q_n of finite propagation idempotents tending in norm to q. Then some q_n is similar to q; hence it is a ghost (because the ghosts form an ideal); hence it is compact (because finite propagation ghosts are compact); hence q is compact also. This is a contradiction.

Here is a brief sketch of Yu's method for proving the Coarse BC conjecture when X can be coarsely embedded in a (possibly infinite-dimensional) Euclidean space $\mathcal{E}$.
The main idea can still be seen when $\mathcal{E}$ is a finite dimensional space. Let $i\colon X \to \mathcal{E}$ be a coarse embedding.
Recall that $C^*(X)$ can be thought of as generated by matrices (with entries in the compact operators) over $X \times X$, having controlled support.

Definition

Let $C^*(X;\mathcal{E})$ denote the C^*-algebra generated by matrices with controlled support over $X \times X$, having values in $C_0(\mathcal{E}) \otimes \mathfrak{K}$, and satisfying the additional support condition that there is a constant $R > 0$ such that the matrix entry $f_{x,y}$ at $(x,y) \in X \times X$ is supported within $B(i(x);R) \subseteq \mathcal{E}$.

Remark

We would get the same algebra if we required $f_{x,y}$ to be supported within $B(i(y);R)$ instead.

Now Yu proves

1. There is a homomorphism

$$K_*(C^*(X)) \to K_*(C^*(X;\mathcal{E}))$$

given by “asymptotic product with the Bott generator.”

2. The group $K_*(C^*(X;\mathcal{E}))$ is isomorphic to the left side of the Coarse Baum-Connes conjecture, and the homomorphism of (1) above inverts the assembly map.

Let X be a coarse space and suppose that a group Γ acts, preserving the coarse structure.
We can define a notion of (Γ, X)-module: this is an X-module which is also equipped with a unitary representation π of Γ, which is compatible in the natural way with the representation $\rho\colon C_0(X) \to \mathfrak{B}(H)$ which is part of the X-module structure. That is

$$\pi(\gamma)(\rho(f)\xi) = \rho(f^\gamma)(\pi(\gamma)\xi).$$

Example

If μ is a Γ-invariant measure on X then $H = L^2(X, \mu)$ is an (Γ, X)-module.

We can now define "Γ-versions" of all the algebras that we have mentioned so far. For example

- $C^*_\Gamma(X)$: the norm closure of the Γ-invariant, controlled, locally compact operators;
- $D^*_\Gamma(X)$: the norm closure of the Γ-invariant, controlled, pseudolocal operators;

and so on.

Remark

It is an interesting analytical question whether the items "Γ-invariant" and "norm closure" can be interchanged in these definitions.

In this talk we shall be interested mainly in the example where X is the universal cover of a compact manifold M, and $\Gamma = \pi_1(M)$. In that case we have:

Lemma

The algebra $C^*_\Gamma(X)$ is Morita equivalent to (and therefore has the same K-theory as) the reduced group C^*-algebra $C^*_r(\Gamma)$.

Proof.

The space of compactly supported elements of H can be completed to a Hilbert $C^*_r(\Gamma)$-module, and $C^*_\Gamma(X)$ is precisely the algebra of compact Hilbert module operators on it. □

The ‘Paschke duality’ theorem is also valid and takes the following form:

Lemma

We have $K_{i+1}(D^*_\Gamma(X)/C^*_\Gamma(X)) = K_i(M)$.

The boundary map $K_{i+1}(D^*_\Gamma(X)/C^*_\Gamma(X)) \to K_i(C^*_\Gamma(X))$ in the above exact sequence now can be identified with an assembly map

$$K_i(M) \longrightarrow K_i(C^*_r(\pi_1(M))).$$

In fact, this is the most classical case of the assembly map.

Example

Our example of the partitioned manifolds index theorem (lecture 3), using infinite cyclic covers, fits into this context.

I want to describe some joint work with Nigel Higson, see Mapping surgery to analysis I, II, III, K-Theory **33**(2005), 277–346. The initial aim of our work was to obtain the "best possible" understanding of the following

Theorem (Mischenko, Kasparov, Kaminker-Miller, Hilsum-Skandalis, etc)

The higher analytic index of the signature operator is a homotopy invariant. That is, if $h\colon M' \to M$ *is an (orientation-preserving) homotopy equivalence of compact oriented manifolds, and* D, D' *are the signature operators of* M, M' *respectively, then* $A[D] = A[D']$ *in* $K_*(C^*_r(\Gamma))$*, where* $\Gamma = \pi_1(M) = \pi_1(M')$*.*

- What is meant by “best possible”?
- Rewrite the theorem to say that

$$h_*[D'] - [D] \in \mathrm{Ker}(A).$$

- We should like to embed the assembly map in the long exact sequence

$$\dots K_{i+1}(D^*_\Gamma(X)) \longrightarrow K_i(M) \xrightarrow{A} K_i(C^*_r(\Gamma)) \dots$$

- and then say that $h_*[D'] - [D]$ comes from some specific element of the ‘analytic structure set’ $K_{i+1}(C^*_r(D^*_\Gamma(X)))$.

Surgery theory

Such an exact sequence is suggested by surgery theory.

- The fundamental object considered in surgery theory is the structure set $\mathcal{S}(Q)$, which is the collection of equivalence classes of homotopy manifold structures on Q.
- Such a structure is just a homotopy equivalence from a manifold M to Q.
- Two structures are equivalent if there is a commuting diagram

$$\begin{array}{ccc} M & \longrightarrow & M' \\ & \searrow & \downarrow \\ & & Q \end{array}$$

where the horizontal arrow is a diffeomorphism.

The surgery exact sequence

- Surgery theory (Browder, Novikov, Sullivan, Wall) embeds $\mathcal{S}(Q)$ into an exact sequence

$$\to L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(Q) \to H_n(X;\mathbb{L}) \to L_n(\mathbb{Z}\pi)$$

  where the map from homology to L-theory is an assembly map (a counterpart to A).
- Thus we seek an analytic counterpart to the surgery exact sequence.
- A natural transformation from the topological to the analytic versions of the surgery exact sequence will give our "best possible reason" for the homotopy invariance theorem.

An example: positive scalar curvature metrics

Consider the example of a compact spin manifold M.

- Analogy leads us to expect that a positive scalar curvature metric will lead to an ‘analytic structure’ in $K_{n+1}(D^*_\Gamma(X))$ which maps to the homology class of the Dirac operator.
- The vanishing of the higher index will then be a consequence of exactness in the analytic surgery sequence.
- Such a ‘structure’ is given by the K-theory class of $F = f(D)$, where f is a normalizing function which is equal to ± 1 outside a small neighborhood of 0 not meeting the spectrum of $\widetilde{D}$.

The case of surgery

In "Mapping surgery to analysis" the main point is to construct a similar structure invariant associated to a homotopy equivalence of closed manifolds.
The construction uses coarse-geometric index theory.
It gives a commutative diagram

$$\begin{array}{ccccccc}
L_{n+1}(\mathbb{Z}\Gamma) & \longrightarrow & \mathcal{S}(M) & \longrightarrow & H_n(M;\mathbb{L}) & \longrightarrow & L_n(\mathbb{Z}\Gamma) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n+1}(C^*_r\Gamma)\otimes\mathbb{Z}[\frac{1}{2}] & \longrightarrow & K_{n+1}(D^*_\Gamma(X))\otimes\mathbb{Z}[\frac{1}{2}] & \longrightarrow & K_n(M)\otimes\mathbb{Z}[\frac{1}{2}] & \longrightarrow & K_n(C^*_r\Gamma)\otimes\mathbb{Z}[\frac{1}{2}]
\end{array}$$

Sketch of the construction

It uses two key geometric ideas (which come from classical surgery).

1. One can define the signature, not just for a manifold, but for a (suitable notion of) Poincaré duality space.
2. One can detect whether a Poincaré duality space is in fact a manifold by seeing whether the signature is actually the index of an elliptic operator, i.e. is in the image of the assembly map.

Also involved is

Lemma

Let X be a locally compact space, and suppose that is given the continuously controlled coarse structure coming from a compactification $\overline{X}$. Let H be an $\overline{X}$-module, which we also regard as an X-module. Then there is an inclusion of C^*-algebras

$$D^*(X; H) \subseteq D^*(\overline{X}; H).$$

Proof.

Use Kasparov's Lemma (which characterizes pseudolocal operators as those for which $\rho(f)T\rho(g)$ is compact whenever f and g have disjoint supports) to show that a $T \in D^*(X; H)$ is $\overline{X}$-pseudolocal. □

- The structure invariant is now obtained by taking a homotopy equivalence $h\colon M' \to M$, forming the mapping cylinder W, and regarding this as continuously controlled over the compactification with boundary $M' \sqcup M$.
- Apply the sequence of maps

$$K(C^*(W)) \to K(D^*(W)) \to K(D^*(\overline{W})) \to K(D^*(M)).$$

- Exactness in the coarse assembly sequence shows that this vanishes if h is a diffeomorphism.
- This can all be done equivariantly.

The following slides represent work in progress with Higson and J. Kaminker.

The multisignature in surgery

Wall invented a technique to detect the cokernel of the assembly map. In its analytic version it goes as follows. Let $\rho\colon \Gamma \to U(n)$ be a unitary representation. It induces $\rho_*\colon C^*_r(\Gamma) \to M_n(\mathbb{C})$ (we assume Γ is amenable for simplicity) and thus $K_0(C^*_r(\Gamma)) \to \mathbb{Z}$.

Theorem

If $x \in K_0(C^*_r(\Gamma))$ belongs to the image of the assembly map, then the rational number $n^{-1}\rho_*(x)$ is independent of the choice of ρ.

Consequently, from any pair of equidimensional representations one obtains a detector for Coker μ.

The difference trace and the multisignature

We will give an explanation for this detection theorem in terms of the analytic surgery sequence.

The representation ρ has a character χ_ρ, which is a trace on $C^*_r(\Gamma)$: $\chi_\rho(g) = \mathrm{Tr}\,\rho(g)$. Via Morita equivalence one can think of χ_ρ as a densely defined trace on $C^*_\Gamma(X)$.

Lemma

For equidimensional representations ρ_1, ρ_2 the difference $\chi_{\rho_1} - \chi_{\rho_2}$ extends to a densely defined trace on $D^*_\Gamma(X)$.

The detection theorem now follows from exactness in the surgery sequence.

The relative η invariant mod $\mathbb{Z}$

- Let D be selfadjoint elliptic on an odd dimensional manifold.
- The eta function is the sum $\sum \mathrm{Sign}(\lambda)\lambda^{-s}$ taken over nonzero eigenvalues of D.
- Analytic continuation defines $\eta(0)$, the eta invariant of D.
- If ρ_1, ρ_2 are equidimensional representations, the difference of eta invariants with coefficients in the corresponding flat bundle defines a homomorphism

$$K_1(M) \to \mathbb{R}/\mathbb{Z}.$$

The main commutative diagram

The constructions of the preceding slides fit together into a commutative diagram

$$\begin{array}{ccccc} K_0(C^*_r(\Gamma)) & \longrightarrow & K_0(D^*_\Gamma(X)) & \longrightarrow & K_1(M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array}$$

where

- The first vertical map is the difference of the maps induced on K-theory by the characters of ρ_1, ρ_2,
- The second vertical map is induced by the difference trace on $D^*_\Gamma(X)$
- The third vertical map is the relative eta invariant.

Noncommutative spectral sections and eta

- The map $K_0(D^*_\Gamma(X)) \to \mathbb{R}$ also has an eta invariant interpretation.
- The $\mathbb{Z}$ ambiguity in the eta invariant comes from the choice of where to split the spectrum. To make such choices consistently we need a noncommutative spectral section (F. Wu, et al).
- Such a spectral section just 'is' an element of $K_0(D^*_\Gamma(X))$ which lifts the homology class of the operator under consideration.
- The real-valued relative eta invariant associated to a noncommutative spectral section is given by the difference trace.

A rigidity theorem for the Dirac operator

Theorem

Suppose that M is a compact odd-dimensional spin manifold of positive scalar curvature, and that $\Gamma = \pi_1(M)$ is torsion-free and satisfies the Baum–Connes conjecture. Then the relative eta invariants of the Dirac operator associated to any pair ρ_1, ρ_2 of equidimensional representations are zero.

Proof.

Consider the diagram

$$\begin{array}{ccc} K_0(D^*_\Gamma(X)) & \longrightarrow & K_0(D^*_\Gamma(B\Gamma)) \\ & \searrow & \downarrow \\ & & \mathbb{R} \end{array}$$

□