Coarse Index Theory
Lecture 4

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Outline

1. K-homology
2. The assembly map
3. Coarse homotopy
4. A generalized conjecture

K-homology

At the end of the last lecture we saw that coarse K-theory for open cones is a generalized homology theory. In fact it turns out to be *K-homology* — the homology theory dual to K-theory.

We'll begin this lecture by reviewing the definition and properties of *K*-homology.

Let X be a locally compact space and let H be an X-module.

Definition

Let $\Psi_{-1}(X; H)$ denote the C^*-algebra of locally compact operators on the X-module H.

Reminder: an operator T is *locally compact* if $T\rho(f)$ and $\rho(f)T$ are compact, for every $f \in C_0(X)$. Here ρ is the representation defining the X-module structure of H.

Pseudolocal operators

Definition

We will say that a bounded operator T on H is *pseudolocal* if $T\rho(f) - \rho(f)T$ is compact for all $f \in C_0(X)$.

We denote the C^*-algebra of pseudolocal operators by $\Psi_0(X; H)$. It is easy to see that Ψ_{-1} is an ideal in Ψ_0.

Lemma

(Kasparov) An operator T is pseudolocal if and only if $\rho(f)T\rho(g)$ is compact whenever f and g have disjoint supports.

The algebras Ψ depend on the choice of X-module, just as $C^*(X)$ does. To define their K-theory we shall use the same colimit construction as we do in the case of $C^*(X)$.

Definition

The *K-homology* groups of X are defined by

$$K_i(X) := K_{i+1}(\Psi_0(X)/\Psi_{-1}(X)).$$

(This is sometimes referred to as the Paschke duality theorem. However, with our colimit interpretation of K-theory, it becomes almost a tautology.)
Kasparov (following a suggestion of Atiyah) proved that the K-homology groups do constitute a generalized homology theory — they are homotopy invariant and have exact sequences.

Example

Let X be a point. Then $\Psi_0(X) = \mathfrak{B}(H)$ and $\Psi_{-1}(X) = \mathfrak{K}(H)$. Thus the quotient is the Calkin algebra, and

$$K_i(X) = K_{i+1}(\mathfrak{B}/\mathfrak{K}) = \begin{cases} \mathbb{Z} & (i = 0) \\ 0 & (i = 1) \end{cases}.$$

This 'explains' the dimension shift in the definition of K-homology.

- Suppose that X is a manifold. Then every elliptic operator D on X gives rise to a K-homology class.
- If D is essentially self-adjoint we can construct this as $[\chi(D)]$, just as in our discussion of the coarse index but neglecting the coarse structure.
- **Remark:** This version of K-homology is a *locally finite homology theory* — it pairs with K-cohomology *with compact supports*.

Notice that the Ψ algebras make no use of any coarse structure.

Definition

Let X be a proper coarse space and H an X-module. Define $D^*(X; H)$ to be the C^*-algebra generated by the controlled, pseudolocal operators on H.

Lemma

Forgetting the coarse structure induces an isomorphism of C^*-algebras

$$D^*(X)/C^*(X) \cong \Psi_0(X)/\Psi_{-1}(X).$$

We will outline the proof.

- Choose a locally finite partition of unity ϕ_i subordinate to a uniformly bounded open cover of X. (This uses properness.)
- The operation

$$\Phi(T) = \sum_i \rho(\phi_i^{1/2}) T \rho(\phi_i^{1/2})$$

  defines a positive linear map $\mathfrak{B}(H) \to \mathfrak{B}(H)$, and its range consists of controlled operators.

Observe that if T is pseudolocal, then

$$\Phi(T) - T = \sum_i [\rho(\phi_i^{1/2}), T] \rho(\phi_i^{1/2})$$

is locally compact.

It now suffices to show

1. $\Psi_0 = D^* + \Psi_{-1}$
2. $C^* = D^* \cap \Psi_{-1}$

Item (1) follows directly from the last observation on the previous slide. Indeed, if T is pseudolocal then $\Phi(T) - T$ is locally compact (belongs to Ψ_{-1}) and $\Phi(T)$ is pseudolocal and of finite propagation, so belongs to $D^*(X)$.

As for item (2), let $T \in D^* \cap \Psi_{-1}$. Then T is locally compact and there is a sequence T_n of finite propagation, pseudolocal operators tending to T. The operators $T_n - \Phi(T_n)$ are locally compact and of finite propagation, so their limit $T - \Phi(T)$ belongs to $C^*(X)$.

But $\Phi(T)$ is of finite propagation, and it is locally compact because T is, so belongs to $C^*(X)$. Thus

$$T = (T - \Phi(T)) + \Phi(T)$$

belongs to $C^*(X)$ also.

Consider the short exact sequence

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0$$

of C^*-algebras. The associated K-theory long exact sequence contains

$$K_{i+1}(D^*(X)/C^*(X)) \to K_i(C^*(X)).$$

But $K_{i+1}(D^*(X)/C^*(X)) = K_{i+1}(\Psi_0(X)/\Psi_{-1}(X)) = K_i(X)$ by the lemma we just proved. Thus we get a map

$$A\colon K_i(X) \to K_i(C^*(X))$$

from K-homology to the K-theory of the coarse C^*-algebra.

Definition

A is called the (coarse) *assembly map*.

The coarse assembly map takes the K-homology class of an elliptic operator to its coarse index. Thus, the basic problem of coarse index theory can be reformulated as: *Compute the coarse assembly map.*

Theorem

If $X = \mathcal{O}(Z)$ is the open cone on a finite simplicial complex Z, then the coarse assembly map for X is an isomorphism.

We have already proved this, essentially. The proof is an inductive argument simplex-by-simplex on Z, using the Mayer-Vietoris sequences in homology and coarse K-theory, and the five lemma.

Could the assembly map *always* be an isomorphism? Clearly not, because the left hand side depends on the small scale (topological) structure of X only and the right hand side depends on the large scale (coarse) structure only. For cones, the small and large scale structures exactly match — that is why the assembly map is an isomorphism.

In the next definition, let E be a controlled set in a coarse space X. A map $Y \to X$ will be called an *E-map* if its range lies within some ball E_x.

Definition

Let X be a proper coarse space. Say that X is *uniformly contractible* if for every n and every controlled E, there is a controlled F such that every continuous E-map $S^{n-1} \to X$ extends to a continuous F-map $D^n \to X$.

The coarse Baum–Connes conjecture

Observe that an open cone on a finite complex is uniformly contractible.

Most computations of coarse K-theory are organized around versions of

Conjecture

If X is uniformly contractible and of bounded geometry, then the coarse assembly map for X is an isomorphism.

Examples (Dranishnikov-Ferry-Weinberger) show that the bounded geometry condition is necessary.

- A natural test case for the coarse Baum–Connes conjecture: consider complete simply-connected Riemannian manifolds of non-positive curvature.
- The properties of geodesics (i.e., the Cartan-Hadamard theorem) tell us that such a manifold is uniformly contractible. In fact, each metric ball is contractible.
- The logarithm map $M \to T_{x_0}M$ is a coarse map to Euclidean space, but it is not a coarse equivalence. So it does not immediately gives rise to a computation of $K_*(C^*(M))$.
- We need a more flexible notion of equivalence.

Coarse homotopy

Let X and Y be proper coarse spaces.

Definition

A proper and continuous map $H\colon X \times [0,1] \to Y$ is a *coarse homotopy* if the associated family of maps $h_t\colon X \to Y$ is "equi-coarse": that is, for every controlled set $E \subseteq X \times X$, the union $\bigcup_t (h_t \times h_t)(E) \subseteq Y \times Y$ is controlled.

This is *strictly weaker* than the assertion that H is a coarse map (when $X \times [0,1]$ is given a *product* coarse structure). (It is equivalent to the assertion that H is coarse from another, weaker coarse structure.) The "tracks" $t \mapsto h_t(x)$ can be arbitrarily long.

Key example

Consider $X = \mathbb{R}^n$ and let (r, θ) be polar coordinates ($\theta \in S^{n-1}$). Let ϕ be any Lipschitz function $\mathbb{R}^+ \to \mathbb{R}^+$ which is increasing and tends to ∞ (e.g. $\phi(r) = \log^+ \log^+ \log^+ r$). Then $f\colon (r, \theta) \mapsto (\phi(r), \theta)$ defines a coarse map. Moreover,

$$F\colon (r, \theta, t) \mapsto ((1-t)\phi(r) + tr, \theta)$$

gives a coarse homotopy between f and the identity.
Let Y be a complete, simply-connected, nonpositively curved Riemannian manifold. Let $\ell\colon Y \to X$ be the logarithm map and let $g\colon X \to Y$ be $\exp \circ f$ where f is as above. By suitable choice of ϕ we can make g coarse. Then g and ℓ are inverses up to coarse homotopy.
Thus Y is coarsely homotopy equivalent to Euclidean space.

Theorem

Coarsely homotopy equivalent coarse maps $X \to Y$ induce the same homomorphism $K_*(C^*(X)) \to K_*(C^*(Y))$.

This theorem was proved by Higson–R. However, recently Viet–Trung Luu has found a more elegant proof.

It involves replacing the Hilbert spaces in the definition of $C^*(X)$ by *Hilbert A-modules*. A Hilbert A-module is just “a Hilbert space in which the scalars are taken from the C^*-algebra A”. Using this notion one can define $C^*(X; A)$, the coarse C^*-algebra with coefficients in A.

Luu shows that a coarse homotopy H defines a map $H_*\colon K_*(C^*(X)) \to K_*(C^*(Y; C[0,1]))$. Then he proves

Proposition

There is a natural 'product'

$$K_*(C^*(Y; C(Z))) \otimes K_*(Z) \to K_*(C^*(Y)).$$

The group $K_*(Z)$ appearing here is the K-homology of Z. In the case $Z = [0,1]$, let i_0, i_1 be the two natural inclusions of a point into Z.Then one can see that the composite

$$K_*(C^*(X)) \xrightarrow{H_*} K_*(C^*(Y; C[0,1])) \xrightarrow{\otimes i_{t*}(1)} K_*(C^*(Y))$$

is equal to $H(\cdot, t)_*$. But $i_{0*}(1) = i_{1*}(1)$ by the homotopy invariance of K-homology. It follows that $H(\cdot, 0)_* = H(\cdot, 1)_*$

Using coarse homotopy one can show that the coarse Baum–Connes conjecture is true for Hadamard manifolds M.

Further progress has come from a different approach. First, let us reformulate the conjecture in a way that does not use uniform contractibility. In fact it makes most sense to formulate this new version for *discrete* bounded geometry spaces.

Definition

Let X be a discrete bg coarse space and let E be a controlled set. The *Rips complex* $P_E(X)$ is the simplicial complex whose vertices are the points of X, and such that $x_0, \ldots, x_p$ span a p-simplex iff $\{x_0, \ldots, x_p\}$ is contained in some E-ball.

If $E \subseteq E'$ then $P_E(X)$ is a subcomplex of $P_{E'}(X)$. Therefore, the homology groups $K_*(P_E(X))$ form a direct system.

Definition

The *coarse K-homology* of X, denoted $KX_*(X)$, is the direct limit

$$\lim_{\underset{E}{\rightarrow}} K_*(P_E(X)).$$

It can be shown that, if there exists a uniformly contractible space EX coarsely equivalent to X, then $KX_*(X) = K_*(EX)$.

Passing to the limit we obtain the *coarse assembly map*

$$KX_*(X) \to K_*(C^*(X))$$

for any (discrete, but this condition can be removed) coarse space X.

It is natural to conjecture

Conjecture

The coarse assembly map is always an isomorphism (?)

The work of Guoliang Yu has shown that this conjecture is true in a very many cases (e.g., whenever X can be coarsely embedded in a Hilbert space). However, it is *false* in general! (Higson 2000)

Box spaces

Let G be a finitely generated discrete group.

Definition

G is *residually finite* if there exists a sequence G_n of finite index normal subgroups of G with $\bigcap G_n = \{1\}$.

If G is residually finite, we define the *box space* $\Box G$ to be the coarse disjoint union $\bigsqcup G/G_n$. Notice that G acts on each G/G_n and thus on $\Box G$; we define the coarse structure on $\Box G$ by requiring that E is controlled if and only if there is a finite subset F of G such that, for each $(x, y) \in E$, there is $g \in F$ such that $x = gy$.

Definition

A *translation* of a coarse space X is a bijection $X \to X$ whose graph is controlled.

Proposition

Let G be a (discrete) group that acts on a (discrete) coarse space X by translations. There is an induced $*$-homomorphism

$$C^*_{\max}(G) \to C^*(X).$$

It is important that we use the *maximal* C^*-algebra here. In particular, we get a homomorphism $C^*_{\max}(G) \to C^*(\Box G)$.

Property T

Definition

Let G be a discrete group. G has *property T* if there exists a projection $p \in C^*_{\max}(G)$ that has the following property: for every representation $\rho\colon G \to U(H)$, the image $\rho(p) \in \mathfrak{B}(H)$ is the orthogonal projection onto the subspace of G-invariant vectors.

The definition is due to Kazhdan, and p is called the *Kazhdan projection*. Obviously every finite group has property T, but there exist infinite examples also (this is rather surprising).

Let G be an infinite, residually finite, property T group (e.g. $G = SL(3, \mathbb{Z})$) and let $X = \square G$. Let $q \in C^*(X)$ be the image of the Kazhdan projection p under the canonical homomorphism $C^*_{\max}(G) \to C^*(X)$.

We know that q is the orthogonal projection onto the G-invariant functions, i.e. those functions constant on each coarse component of X. In particular, q is a projection with infinite-dimensional range.

Remark

q is *not a controlled operator*. Property T tells us that it is a limit of controlled operators, but this is mysterious!

Higson showed that (under appropriate hypotheses, fulfilled e.g. in the case of $SL(3, \mathbb{Z})$) the K-theory class $[q] \in K_0(C^*(X))$ does not belong to the image of the coarse assembly map. We'll prove a related but weaker result, namely that q is *not* a limit of finite propagation *idempotents*. (The relationship is that any index can, as we know, be represented by an idempotent of finite propagation.)

Definition

(Yu) Let X be a discrete coarse space. An operator $T \in C^*(X)$ is a *ghost* if its matrix entries tend to 0.

The ghosts form an ideal in $C^*(X)$ which is (possibly) not 'geometric', i.e. not of the form I_Y.

Lemma

Every compact operator is a ghost, and every finite propagation ghost is compact.

The element $q \in C^*(\Box G)$ coming from the Kazhdan projection is a ghost but *not* compact (it is a projection with infinite-dimensional range).

Suppose that there is a sequence q_n of finite propagation idempotents tending in norm to q. Then some q_n is similar to q; hence it is a ghost (because the ghosts form an ideal); hence it is compact (because finite propagation ghosts are compact); hence q is compact also. This is a contradiction.