Coarse Index Theory
Lecture 3

John Roe

Penn State University

May 2006

Outline

1. Operator algebras associated to coarse structure
2. Index theory for partitioned manifolds
    - Application to bordism
    - Application to positive scalar curvature
    - Proof of the partitioned index theorem
3. *K*-theory calculations
    - Colimits
    - The Mayer-Vietoris sequence
    - A vanishing theorem

Compatibility with topology

Definition

Let X be a Hausdorff space. We say that a coarse structure on X is *proper* if

1. There is a controlled neighborhood of the diagonal, and
2. Every bounded subset of X is relatively compact.

Notice that X must be locally compact.

Theorem

Let X be a connected topological space provided with a proper coarse structure. Then X is coarsely connected. A subset of X is bounded if and only if it is relatively compact. Moreover, every controlled subset of $X \times X$ is proper.

Examples

Example

The (bounded) coarse structure on a metric space is proper if the metric is a *proper metric* (closed bounded sets are compact).

Example

The continuously controlled coarse structure defined by a metrizable compactification is always proper.

When dealing with proper coarse spaces it is natural to consider the subcategory whose morphisms are *continuous* coarse maps.

Geometric Hilbert spaces

Let X be a locally compact Hausdorff space.

Definition

A geometric X-module is a Hilbert space H, equipped with a representation $\rho\colon C_0(X) \to \mathfrak{B}(H)$.

For example, $L^2(X, \mu)$ (relative to a Borel measure μ) is a geometric X-module.

Definition

Let H be a geometric X-module. The *support* of $\xi \in H$ is the complement of the set of $x \in X$ having the following property: there is a neighborhood U of x in X such that $\rho(C_0(U))$ annihilates ξ.

Properties of supports

It is easy to see that the support is a closed set and

$$\mathrm{Support}(\xi + \xi') \subseteq \mathrm{Support}(\xi) \cup \mathrm{Support}(\xi').$$

Moreover, $\mathrm{Support}(\rho(f)\xi) \subseteq \mathrm{Support}(f) \cap \mathrm{Support}(\xi)$. Suppose that X is a proper coarse space and H is a geometric X-module. Let $T \in \mathfrak{B}(H)$.

Definition

We say that T is *controlled* if there is a controlled set E such that for all $\xi \in M$,

$$\mathrm{Support}(T\xi) \cup \mathrm{Support}(T^*\xi) \subseteq E[\mathrm{Support}(\xi)].$$

Here the notation $E[K]$ refers to $\{x : \exists y \in K,\ (x,y) \in E\}$.

Theorem

The controlled operators form a $*$-algebra of operators on H.

Definition

An operator T on a geometric X-module H is *locally compact* if $\rho(f)T$ and $T\rho(f)$ are compact whenever $f \in C_0(X)$.

Let X be a proper coarse space and let H be a geometric X-module.

Definition

The *translation algebra* $C^*(X; H)$ is the C^*-algebra generated by the controlled, locally compact operators on H.

Functoriality of translation algebras

Let $f\colon X \to Y$ be a coarse and continuous map of proper coarse spaces. Then f induces a map $C_0(Y) \to C_0(X)$, which in turn makes every X-module H into a Y-module $f_*(H)$.

Theorem

In the above situation there is a natural map of C^*-algebras

$$f_*\colon C^*(X; H) \to C^*(Y; f_* H).$$

It is not really necessary to restrict attention to continuous maps here. With some small extra complication we can make $C^*(X)$ functorial under general coarse maps.

An example of the coarse index

- Last time we considered elliptic operators (self-adjoint, first order, finite propagation speed) on complete Riemannian manifolds M.
- We saw that such an operator has a *coarse index* in $K(C^*(M))$, where $C^*(M)$ is the algebra associated to the bounded coarse structure of M.
- In this lecture we shall give techniques for computing these indices and K-theory groups.
- We begin with an easy example.

Let M be a complete Riemannian manifold.

Definition

A *partition* of M is a compact codimension-one submanifold N of M that disconnects M, together with an orientation for its normal bundle.

For example, the origin $\{0\}$ is a partition of $\mathbb{R}$. This is the 'universal example': any partition of a manifold is pulled back from this partition of $\mathbb{R}$ via a proper map.

We are going to use a partition N of a manifold M to construct a functional $\tau_N\colon K_1(C^*(M)) \to \mathbb{Z}$.

The construction is inspired by the classical theory of *Wiener-Hopf operators.* Such an operator is a composite $W_f = PC_f$, where C_f is the operator on $L^2(\mathbb{R})$ defined by convolution with an L^1 function f, and P is the orthogonal projection $L^2(\mathbb{R}) \to L^2(\mathbb{R}^+)$.

Analogously, for $a \in C^*(M)$ define W_a to be the operator $P \circ a \circ i$ on $L^2(M^+)$: here $i\colon L^2(M^+) \to L^2(M)$ is the inclusion, $P\colon L^2(M) \to L^2(M^+)$ is the projection, and a acts on $L^2(M)$.

Lemma

The assignment $a \mapsto W_a$ is a $*$-homomorphism modulo compacts: that is, $W_{aa'} - W_a W_{a'}$ and $W_a^* - W_{a^*}$ are compact.

Proof.

It suffices to show $[P, a]$ is compact, and we may assume that a is locally compact and of finite propagation. Then $[P, a]$ is locally compact and supported in a bounded neighborhood of N, hence it is compact. □

We therefore may pass to the Calkin algebra and obtain a $*$-homomorphism

$$C^*(M) \to \mathfrak{B}(L^2(M^+))/\mathfrak{K}(L^2(M^+)).$$

On the K-theory level this $*$-homomorphism induces a map

$$K_1(C^*(M)) \to K_1(\mathfrak{B}/\mathfrak{K}) \to K_0(\mathfrak{K}) = \mathbb{Z}$$

which is our homomorphism τ_N.

Now the index question is this: if D is an elliptic operator on M, what is $\tau_N(\mathrm{Index}(D))$? For this question to be significant, M must be *odd*-dimensional.

Theorem

One has $\tau_N(\mathrm{Index}\ D) = \mathrm{Index}\ D_N$, where the index on the right is an ordinary (Atiyah-Singer) index of the 'restricted operator' D_N on the compact manifold N.

For example, the ‘restriction’ of the signature operator (for M odd-dimensional) is again the signature operator.
Let N and N' be partitions that are *bordant*, meaning that $N \cup N'$ bounds a *compact* portion of M. Then the corresponding Wiener-Hopf operators W_a, W'_a are equal modulo compacts and so define the same homomorphism to the Calkin algebra. Therefore $\tau_N = \tau_{N'}$.

Corollary

The signature (of a compact manifold) is an oriented bordism invariant.

This is a proof, using analysis on non-compact manifolds, of a classical theorem of geometric topology.

Corollary

If Index $D_N \neq 0$*, then the spectrum of* D *is the whole real line. In particular,* D *is not invertible.*

Let X be a compact manifold and $c \in H^1(X; \mathbb{Z})$. By standard algebraic topology, c is pulled back from the generator of $H^1(S^1)$ via a *classifying map* $X \to S^1$.

The corresponding infinite cyclic cover of X is a partitioned manifold. The Atiyah-Singer index theorem and Poincaré duality give

$$\text{Index}\, D_N = \langle \mathfrak{I}_D \smile c, [X] \rangle$$

where $\mathfrak{I}_D$ is the 'index form' of the operator D.

We can apply this to the Dirac operator to give conditions under which X has no metric of positive scalar curvature.

One can prove the theorem in three stages.

1. Direct calculation (using separation of variables) for the product case $M = N \times \mathbb{R}$.
2. Homotopy invariance to allow us to make the metric a product in a tubular neighborhood of N.
3. Localization principle: the index depends only on the geometry of a neighborhood of N (uses the fact that the index can be represented by operators of *arbitrarily small* propagation).

Defining $K(C^*(X))$

Let X be a coarse space. The algebra $C^*(X)$, and therefore its K-theory, apparently depends on the choice of X-module used to define the algebra.

One can eliminate this dependence in a number of different ways.

1. Prove a "Voiculescu-type" theorem saying that any "sufficiently large" X-module will absorb any other one.
2. Consider all X-modules at once (categorical technique).

We shall do the second of these.

Let I be a small category and let Ab be the category of abelian groups. Let $F\colon I \to Ab$ be a (covariant) functor.

Definition

A *compatible F system* is a natural transformation from F to a constant functor (with values in Ab). That is, it is an abelian group G together with homomorphisms $\alpha_i\colon F(i) \to G$, for each object $i \in I$, such that for each morphism $\lambda\colon i \to j$ in I the diagram

$$\begin{tikzcd}
F(i) \arrow[r, "F(\lambda)"] \arrow[dr, "\alpha_i"] & F(j) \arrow[d, "\alpha_j"] \\
 & G
\end{tikzcd}$$

commutes.

Definition

A *colimit* of the functor F is a universal compatible F system.

That is to say, a colimit G is a compatible F system, and if H is another compatible F system there is a unique homomorphism $G \to H$ making the obvious diagrams commute.

Lemma

Colimits always exist (in the category of abelian groups).

To apply this machinery to $C^*(X)$, let I be the category whose objects are X-modules and whose morphisms are (injective) isometries of Hilbert space preserving the X-module structure. (For set-theoretic reasons it is preferable to consider X-module structures on some *fixed* Hilbert space; this makes no essential difference.)

Then $H \mapsto K_i(C^*(X;H))$ becomes a covariant functor $I \to Ab$, and we shall define $K_i(C^*(X))$ to be the colimit of this functor.

We now have a functor $X \mapsto K_*(C^*(X))$ defined on the category of coarse spaces, and we want to compute it. The first computational result is a *Mayer-Vietoris sequence* due to Higson–R–Yu. Recall that the classical Mayer-Vietoris sequence of algebraic topology allows one to compute the K-theory or cohomology of a space $X = A \cup B$ from the corresponding information about A, B, and $A \cap B$.

Definition

Let X be a coarse space and let $Y \subseteq X$ be a subspace. An operator T is *supported near* Y if there is a controlled set E such that, if $\mathrm{Support}(\xi) \cap E[Y] = \emptyset$, then $T\xi = T^*\xi = 0$.

Denote by I_Y the norm closure of the operators supported near Y; it is a closed ideal in $C^*(X)$.

Example

If Y is bounded, then I_Y is the ideal of compact operators in $C^*(X)$.

Lemma

The K-theory of the ideal I_Y is equal to the K-theory of $C^*(Y)$.

Proof.

By construction, I_Y is a C^*-algebraic direct limit $\lim_E C^*(E[Y])$. But each $E[Y]$ is coarsely equivalent to Y itself, so the inclusion $Y \to E[Y]$ induces an isomorphism on coarse K-theory. Passing to the direct limit we obtain the result. ☐

Suppose $X = Y_1 \cup Y_2$ is the union of two subspaces. We ask: What is the relationship between $I_{Y_1} \cap I_{Y_2}$ and $I_{Y_1 \cap Y_2}$? It is easy to see that these are not the same in general.

Example

Let X be the set of points $(x, y) \in \mathbb{R}^2$ such that either $x > 0$ and $y \in \{0, 1\}$, or $x = 0$ and $y \in [0, 1]$; we give it the metric induced from $\mathbb{R}^2$. Let $Y_1 = \{(x, y) \in X : y \leqslant \frac{1}{2}\}$ and let $Y_2 = \{(x, y) \in X : y \geqslant \frac{1}{2}\}$. Then $I_{Y_1} = I_{Y_2} = C^*(X)$, whereas $I_{Y_1 \cap Y_2}$ is just the compact operators.

Definition

The decomposition $X = Y_1 \cup Y_2$ is *coarsely excisive* if for every controlled set E there is a controlled set F such that $E[Y_1] \cap E[Y_2] \subseteq F[Y_1 \cap Y_2]$.

Lemma

If $X = Y_1 \cap Y_2$ *is a coarsely excisive decomposition then* $I_{Y_1} \cap I_{Y_2} = I_{Y_1 \cap Y_2}$ *and* $I_{Y_1} + I_{Y_2} = C^*(X)$*.*

This is because an operator supported near Y_1 and supported near Y_2 must be supported near $Y_1 \cap Y_2$.

By standard techniques of C^*-algebra K-theory, this gives us the Mayer-Vietoris exact sequence

$$\begin{tikzcd}
K_1(C^*(Y_1 \cap Y_2)) \arrow[r] & K_1(C^*(Y_1)) \oplus K_1(C^*(Y_2)) \arrow[r] & K_1(C^*(X)) \arrow[d] \\
K_0(C^*(X)) \arrow[u] & K_0(C^*(Y_1)) \oplus K_0(C^*(Y_2)) \arrow[l] & K_0(C^*(Y_1 \cap Y_2)) \arrow[l]
\end{tikzcd}$$

Theorem

For any X, the space $Y = X \times \mathbb{R}^+$ has $K_*(C^*(Y)) = 0$.

The proof involves an ‘Eilenberg swindle’ argument.
Let p be a projection representing an element of $K_0(C^*(Y; H))$ (say) and let V be the ‘right shift’ operator. Then

$$q = p \oplus \mathrm{Ad}(V)p \oplus \mathrm{Ad}(V^2)p \oplus \cdots$$

is a projection representing an element of $K_0(C^*(Y; \oplus^\infty H))$.
Moreover, $p \oplus \mathrm{Ad}(V)q = q$. It follows that $[p] = 0$ in K-theory

- Together with the Mayer-Vietoris sequence, this allows us to compute $K_i(C^*(\mathbb{R}^n))$. It is $\mathbb{Z}$ if $i \equiv n \bmod 2$, and 0 otherwise.
- The partitioned manifolds index theorem shows that the generator of $K_1(C^*(\mathbb{R}))$ is the coarse index of the operator id/dx on $\mathbb{R}$.

Theorem

The generator of $K_n(C^*(\mathbb{R}^n))$ is the coarse index of the Dirac operator on $\mathbb{R}^n$.

More generally, define the *open cone* functor $\mathcal{O}$ which sends a finite simplicial complex Z (which we think of as embedded in S^{n-1} for some n) to the coarse space $\mathcal{O}Z$ which is the union of all the rays from the origin through points of Z.

Theorem

The functor $Z \mapsto K_*(C^*(\mathcal{O}Z))$ is a reduced generalized homology theory.

This is the first hint of a connection between coarse C^*-algebras and K-homology, which we shall investigate in much more detail next time.