# Index theory and coarse geometry Lecture 2

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# Outline

# Non compact elliptic theory

Coarse index theory begins from the idea of generalizing the index theorem to Riemannian manifolds that are *complete* but *noncompact*. A helpful example to bear in mind is the unit disk in $\mathbb{C}$, equipped with its Poincaré (hyperbolic) metric. It is easy to see that

- The kernel of an elliptic operator $D$ now depends on what global growth conditions we impose (e.g. there exist nonzero bounded harmonic functions on the disk, but no nonzero $L^2$ harmonic functions).
- The kernel may be infinite dimensional, even with the 'nicest' growth conditions (e.g. $L^2$ harmonic 1-forms on the disk.)

We will always work with $L^2$ growth conditions. Let $M$ be a complete Riemannian manifold and let $D$ be a first order differential operator (acting on sections of a hermitian vector bundle $E$). If $\langle Ds, s' \rangle = \langle s, Ds' \rangle$ for all *smooth, compactly supported* sections $s, s'$ we say that $D$ is *formally self-adjoint*.

**Definition**

The *propagation speed* of $D$ is $\sup \|\sigma_D(\xi)\|$, where the supremum is taken over all unit (co)vectors $\xi$ in $T^*M$.

The natural geometric operators (e.g. the de Rham operator $d + d^*$) have unit propagation speed.

**Theorem**

*Let $M$ be complete and $D$ be formally self-adjoint with finite propagation speed. Then $D$ is essentially self-adjoint.*

Let $D$ be as above. The essential self-adjointness of $D$ means that we can apply the spectral theorem to form an operator $f(D)$ for every $f \in C_0(\mathbb{R})$, or indeed for every bounded Borel function $f$ on $\mathbb{R}$.

**Theorem**

*The operator D has locally compact resolvent, in other words the operator $f(D)$ is locally compact for all $f \in C_0(\mathbb{R})$.*

Here an operator $T$ on $L^2(M)$ is said to be *locally compact* if $TM_g$ and $M_g T$ are compact whenever $M$ is the operator of multiplication by some $g \in C_0(M)$.

The resolvents $f(D)$ have another property which has more geometric content. To state it, it is convenient first to study the *wave operators* $e^{itD}$.

**Theorem**

*Let $D$ be a self-adjoint elliptic operator on a complete Riemannian manifold $M$, having finite propagation speed $c$. Then the wave operator $e^{itD}$ has the property*

$$\mathrm{Support}(e^{itD}s) \subseteq N(\mathrm{Support}(s); c|t|).$$

**Proof.**

Energy estimates. □

This result gives information about a general operator of the form $f(D)$ because of the Fourier representation

$$f(D) = \frac{1}{2\pi} \int \hat{f}(t) e^{itD} dt.$$

Thus if $\hat{f}$ is compactly supported, $f(D)$ is an operator of *finite propagation*: there is a constant $R$ such that

$$\text{Support}(f(D)s) \subseteq N(\text{Support}(s); cR).$$

A general $f(D)$ is a limit (in norm) of finite propagation operators.

Let $M$ be a complete Riemannian manifold and let $D$ be a self-adjoint first order differential operator on a bundle $E$ over $M$, having finite propagation speed. Let $A$ denote the $C^*$-algebra which is the norm closure of the finite propagation, locally compact operators on $H = L^2(M; E)$. The results that we have stated prove that $f(D) \in A$ for all $f \in C_0(\mathbb{R})$.

Thus, by the constructions of the previous lecture, there is a *coarse index* $\text{Index}(D) \in K_i(A)$, where $i = 0$ in the graded case and 1 in the ungraded case. The project of coarse index theory is to understand this index, and the group to which it belongs.

- Suppose for example that $M$ is a compact manifold. Then every operator on $L^2(M)$ is of finite propagation, and the locally compact operators are just the compact ones.
- The algebra $A$ is then just the algebra $\mathfrak{K}$ of compact operators, and $K_0(A) = \mathbb{Z}$.
- The coarse index in this case is just the usual Atiyah-Singer index.

# The heat equation method

Continue to consider the example of a compact $M$. There is a proof of the index theorem (due to Atiyah, Bott and Patodi) which fits well with the 'coarsening' idea. This is the *heat equation* proof.

- It is based on the *heat equation*

$$\frac{\partial s}{\partial t} + D^2 s = 0.$$

- The operator $e^{-tD^2}$ can be defined by the functional calculus and provides the *solution operator* to the above equation with prescribed initial data.

The operator $e^{-tD^2}$ is represented by a *smoothing kernel*

$$e^{-tD^2}s(x) = \int k_t(x, y)s(y)dy$$

where $k_t$ is a smooth function that approaches the familiar Gaussian shape as $t \downarrow 0$.

**Definition**

The *trace* of a smoothing operator on a compact manifold can be defined by integrating its kernel over the diagonal.

**Lemma**

*(McKean-Singer) For any $t > 0$ the 'supertrace'*

$$\mathrm{Tr}(\epsilon e^{-tD^2})$$

*exists and is equal to the index of $D$.*

Let us prove the McKean-Singer formula using $K$-theory.

1. The $K$-theory index is $[-S\epsilon S] \ominus [\epsilon]$, where $S = \chi(D) + \epsilon\sqrt{1 - \chi(D)^2}$.
2. Choose $\chi$ so that $1 - \chi(\lambda)^2 = e^{-t\lambda^2}$.
3. The (integer) index is $\mathrm{Tr}(-S\epsilon S - \epsilon)$, and this gives the McKean-Singer formula.

The McKean-Singer formula led to a proof of the index theorem on compact manifolds via an analysis of the asymptotic behavior of the heat kernel $k_t$ as $t \downarrow 0$.On non-compact manifolds, smoothing operators need not be traceable so the proof fails. However, it still leads to the following philosophy:

- For short times $t$, the heat operator $e^{-tD^2}$ represents a local (topological) invariant.
- For long times, it represents a global (analytical) invariant — the projection onto the kernel of $D$.
- Passing from the operator $D$ to its index is a process of passing from local to global — *coarsening*.

We now give axioms which qualitatively model the 'large scale structure' of a space. Compare *topology* which models the *small scale* structure.

### Definition

A *coarse structure* on a set $X$ is a collection $\mathcal{E}$ of subsets of $X \times X$, called the *controlled sets* or *entourages* for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions. A set equipped with a coarse structure is called a *coarse space*.

Here by the *product* of $E, E' \subseteq X \times X$ we mean the set

$$\{(x,z) : \exists y,\ (x,y) \in E,\ (y,z) \in E'\}.$$

These axioms are 'increasingly directed'; contrast Weil's axioms for a uniformity.

# The basic example

## Example

Let $(X, d)$ be a metric space and let $\mathcal{E}$ be the collection of all those subsets $E \subseteq X \times X$ for which the coordinate projection maps $\pi_1, \pi_2 \colon E \to X$ are close; otherwise put, the supremum

$$\sup\{d(x, x') : (x, x') \in E\}$$

is finite. Then $\mathcal{E}$ is a coarse structure. It is called the *bounded coarse structure* associated to the given metric.

# More examples

**Example**

(N. Wright) Again, let $(X, d)$ be a metric space and let $\mathcal{E}$ be the collection of all those subsets $E \subseteq X \times X$ for which the distance function $d$, when restricted to $E$, tends to zero at infinity. Then $\mathcal{E}$ is a coarse structure on $X$, called the $C_0$ *coarse structure* associated to the metric $d$.

In the next result, let $X$ be a locally compact space. We say that a subset $E \subseteq X \times X$ is *proper* if, for every compact $K \subseteq X$, the set

$$\{x : \exists y,\ \{(x, y), (y, x)\} \cap E \neq \emptyset\}$$

is compact.

### Theorem

*Let $X$ be a locally compact Hausdorff space, with a metrizable compactification $\overline{X}$. Let $E \subseteq X \times X$. The following conditions are equivalent.*

1. *The closure $\overline{E}$ of $E$ in $\overline{X} \times \overline{X}$ meets the complement of $X \times X$ only in the diagonal $\Delta_{\partial X} = \{(\omega, \omega) : \omega \in \partial X\}$.*
2. *$E$ is proper, and for every sequence $(x_n, y_n)$ in $E$, if $\{x_n\}$ converges to a point $\omega \in \partial X$, then $\{y_n\}$ also converges to $\omega$.*

*Moreover, the sets $E$ satisfying these equivalent conditions form the controlled sets for a coarse structure on $X$.*

This is called the *continuously controlled coarse structure* associated to the given compactification.

Let $X$ be a coarse space.

## Definition

Let $Y$ be any set. Two maps $f_1, f_2 \colon Y \to X$ are *close* if $\{(f_1(y), f_2(y)) : y \in Y\}$ is a controlled subset of $X \times X$.

- It is easy to see that closeness is an equivalence relation. In fact, the closeness relation completely determines the coarse structure.
- We can say that two *points* of $X$ are close if the corresponding inclusion maps are close. If all pairs of points are close, $X$ is *coarsely connected*.

## Definition

A subset $B$ of $X$ is *bounded* if the inclusion $B \to X$ is close to a constant map. Equivalently, $B$ is bounded if $B \times B$ is controlled.

# Coarse maps

Let $X$ and $Y$ be coarse spaces.

## Definition

A map $f\colon X \to Y$ is *coarse* if

1. Whenever $E \subseteq X \times X$ is controlled, $(f \times f)(E) \subseteq Y \times Y$ is controlled;
2. Whenever $B \subseteq Y$ is bounded, $f^{-1}(B) \subseteq X$ is bounded also.

For metric (bounded) control, the first condition translates to: for every $R > 0$ there is $S > 0$ such that $d_X(x, x') < R$ implies $d_Y(f(x), f(x')) < S$.

# Examples

## Example

The map from the plane to the line sending $(r, \theta)$ to $r$ (polar coordinates) is a coarse map; the map sending $(x, y)$ to $x$ (rectangular coordinates) is not.

## Example

Let $X$ be the plane with its bounded coarse structure and let $Y$ be the plane with the cc structure coming from its radial compactification. The identity map $X \to Y$ is coarse.

## Example

Let $X$ be complete Riemannian, 1-connected, $K \leqslant 0$. The 'logarithm' map $X \to T_{x_0}X$ is coarse (Cartan-Hadamard).

# Coarse equivalence

A *coarse equivalence* is a coarse map that has an inverse up to closeness. More exactly, a coarse map $f\colon X \to Y$ is a coarse equivalence if there is a coarse map $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps on $X$ and $Y$ respectively.

## Example

The inclusion $\mathbb{Z}^n \to \mathbb{R}^n$ is a coarse equivalence. The inclusion of Example 15 (of the metric plane into the cc plane) is *not* a coarse equivalence (exercise: why not?)

There is a *coarse category* of coarse spaces and maps up to coarse equivalence.

## Definition

Let $X$, $Y$ be coarse spaces. A map $i\colon X \to Y$ that is a coarse equivalence onto its image is called a *coarse embedding*.

(Here of course $i(X)$ inherits a coarse structure from $Y$.) In the metric case, a popular alternative definition is the following.

## Lemma

*If $X$ and $Y$ are metric spaces (with bounded coarse structure) then $i\colon X \to Y$ is a coarse embedding iff there are functions $\rho_1, \rho_2\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho_1(t) \to \infty$ as $t \to \infty$ and*

$$\rho_1(d_X(x, x')) \leqslant d_Y(i(x), i(x')) \leqslant \rho_2(d_X(x, x')).$$

A non trivial example of a coarse embedding is the following.

### Example

Let $T$ be a regular tree and let $H$ be the Hilbert space $L^2(T)$. Fix a base point $x_0 \in T$. Then the map which sends $x \in T$ to the characteristic function of the (unique) geodesic segment $[x_0, x]$ is a coarse embedding.

# Bounded geometry

Let $D$ be a set. For a subset $E \subseteq D \times D$ we define the *cross sections*

$$E_x = \{y : (x, y) \in E\}, \quad E^x = \{y : (y, x) \in E\}.$$

A coarse structure on $D$ is a *discrete bounded geometry structure* on $X$ if for every controlled set $E$ there is a constant $C_E$ such that, for each $x \in D$, the cross sections $E^x$ and $E_x$ have at most $C_E$ elements.

## Definition

A coarse space $X$ has *bounded geometry* if it is coarsely equivalent to some discrete bounded geometry coarse space.

# Compatibility with topology

## Definition

Let $X$ be a Hausdorff space. We say that a coarse structure on $X$ is *proper* if

1. There is a controlled neighborhood of the diagonal, and
2. Every bounded subset of $X$ is relatively compact.

Notice that $X$ must be locally compact.

## Theorem

*Let $X$ be a connected topological space provided with a proper coarse structure. Then $X$ is coarsely connected. A subset of $X$ is bounded if and only if it is relatively compact. Moreover, every controlled subset of $X \times X$ is proper.*

# Examples

## Example

The (bounded) coarse structure on a metric space is proper if the metric is a *proper metric* (closed bounded sets are compact).

## Example

The continuously controlled coarse structure defined by a metrizable compactification is always proper.

When dealing with proper coarse spaces it is natural to consider the subcategory whose morphisms are *continuous* coarse maps.

# Geometric Hilbert spaces

Let $X$ be a locally compact Hausdorff space.

**Definition**

A geometric $X$-module is a Hilbert space $H$, equipped with a representation $\rho\colon C_0(X) \to \mathfrak{B}(H)$.

For example, $L^2(X, \mu)$ (relative to a Borel measure $\mu$) is a geometric $X$-module.

**Definition**

Let $H$ be a geometric $X$-module. The *support* of $\xi \in H$ is the complement of the set of $x \in X$ having the following property: there is a neighborhood $U$ of $x$ in $X$ such that $\rho(C_0(U))$ annihilates $\xi$.

# Properties of supports

It is easy to see that the support is a closed set and

$$\mathrm{Support}(\xi + \xi') \subseteq \mathrm{Support}(\xi) \cup \mathrm{Support}(\xi').$$

Moreover, $\mathrm{Support}(\rho(f)\xi) \subseteq \mathrm{Support}(f) \cap \mathrm{Support}(\xi)$.
Suppose that $X$ is a proper coarse space and $H$ is a geometric $X$-module. Let $T \in \mathfrak{B}(H)$.

## Definition

We say that $T$ is *controlled* if there is a controlled set $E$ such that for all $\xi \in M$,

$$\mathrm{Support}(T\xi) \cup \mathrm{Support}(T^*\xi) \subseteq E[\mathrm{Support}(\xi)].$$

Here the notation $E[K]$ refers to $\{x : \exists y \in K,\ (x, y) \in E\}$.

**Theorem**

*The controlled operators form a $*$-algebra of operators on $H$.*

**Definition**

An operator $T$ on a geometric $X$-module $H$ is *locally compact* if $\rho(f)T$ and $T\rho(f)$ are compact whenever $f \in C_0(X)$.

Let $X$ be a proper coarse space and let $H$ be a geometric $X$-module.

**Definition**

The *translation algebra* $C^*(X; H)$ is the $C^*$-algebra generated by the controlled, locally compact operators on $H$.

# Functoriality of translation algebras

Let $f\colon X \to Y$ be a coarse and continuous map of proper coarse spaces. Then $f$ induces a map $C_0(Y) \to C_0(X)$, which in turn makes every $X$-module $H$ into a $Y$-module $f_*(H)$.

**Theorem**

*In the above situation there is a natural map of $C^*$-algebras*

$$f_*\colon C^*(X; H) \to C^*(Y; f_*H).$$

In fact, the map is the identity map!

Our discussion at the beginning of the lecture shows that (suitable) elliptic operators on a complete Riemannian manifold $M$ have indices in the $K$-theory of the translation algebra of $M$. This motivates us to develop techniques for computing this $K$-theory. We shall do that in the next lecture.