Coarse Index Theory
Lecture 1

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May 2006

Outline

The de Rham complex

Let M be a smooth manifold. Recall that a *k-form* on M is a section of the k'th exterior power of the cotangent bundle. The *exterior derivative* is the differential operator

$$d\colon \Omega^k(M) \to \Omega^{k+1}(M)$$

which is uniquely determined by the facts that

- if $f \in \Omega^0(M)$ is a smooth *function*, then

  $$df(X) = X \cdot f$$

  for all vector fields f, and
- d is a derivation of the graded algebra $\Omega^*(M)$.

Lemma

For any form α, $d(d\alpha) = 0$.

This lemma means that the differential forms form a *complex*

$$\Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \cdots$$

called the *de Rham complex*, and we have

Theorem

The cohomology of the de Rham complex is isomorphic to the topologically defined (singular or Cech) cohomology of M.

In particular, if M is a compact manifold, the de Rham cohomology spaces $H^i(M)$ are *finite dimensional*. This is a non-trivial statement about partial differential equations.

From the de Rham complex of a (compact) manifold we can extract two important numerical invariants:

1. The *Euler characteristic* $\chi(M) = \sum(-1)^i \dim H^i(M)$, and
2. (in the oriented case) the *signature*, discussed below.

Poincaré duality and the signature

An orientation on a compact n-manifold M defines an integration functional $\int : H^n(M) \to \mathbb{R}$. The Poincaré duality theorem states

Theorem

The intersection form

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

defines a nondegenerate pairing $H^k(M) \otimes H^{n-k}(M) \to \mathbb{R}$.

In particular if $n = 4k$ then the intersection form gives a symmetric bilinear form on $H^{2k}(M)$. The *signature* of M is by definition the signature of this symmetric bilinear form.

The Hodge theorem

In order to find a realization of de Rham cohomology in terms of analysis, we need to introduce a *Riemannian metric*. This gives rise to L^2 inner products on all spaces of forms.

Theorem

Each de Rham cohomology class on a compact manifold contains a unique element of minimal L^2 norm.

Simple Hilbert space geometry shows that such a minimal element must be *orthogonal* to the range of d, and therefore it must be in the kernel of the Hilbert space adjoint operator d^*. This directs our attention to the *elliptic operator*

$$D = d + d^*$$

on the space of all differential forms.

Consider a *first order linear differential operator* D on sections of a vector bundle E over M.
There is a unique endomorphism-valued homogeneous function σ_D on T^*M with the property that

$$D(fs) = fDs + \sigma_D(df)s$$

for all (smooth) functions f and sections s of E.

Definition

σ_D is called the *symbol* of D.

Example

The symbol of the exterior derivative operator d is the 1-form that takes a covector ξ to the operation ‘exterior multiplication by ξ’.

Definition

The linear differential operator D is called *elliptic* if, for every nonzero ξ, the endomorphism $\sigma_D(\xi)$ is invertible.

For example, the *de Rham operator* $d + d^*$ that we discussed earlier is elliptic.

The basic analytical information about elliptic operators on compact manifolds is contained in the following theorem.

Theorem

Let D be a self-adjoint elliptic operator on a compact manifold. Then D has compact resolvent: for any $f \in C_0(\mathbb{R})$, the operator $f(D)$ (defined by the functional calculus) is compact.

Corollary

The kernel of an elliptic operator is a finite-dimensional vector space.

Together with the Hodge theorem this provides an analytic explanation for the finite dimensionality of de Rham cohomology. (In fact, the basic analytical facts about elliptic operators are needed in the *proof* of the Hodge theorem.)

Gradings

Definition

Let D be an elliptic operator on a bundle E. A *grading* for D is a self-adjoint involution ϵ of E that anticommutes with D ($D\epsilon + \epsilon D = 0$).

In other words, ϵ splits E into two orthogonal subbundles $E_{\pm}$, and D interchanges them.

Definition

The *index* of a graded operator (D, ϵ) is the integer

$$\dim\ker D_{+} - \dim\ker D_{-}$$

where $D_{\pm}$ denotes the restriction of D to an operator $E_{\pm} \to E_{\mp}$.

Let $D = d + d^*$ denote the de Rham operator on a compact Riemannian manifold.

Example

The involution ϵ which equals $(-1)^k$ on the space of k-forms is a grading, and the index of D with respect to this grading is the Euler characteristic.

Let M be oriented and of dimension $4k$. The *Hodge star operator* is a (linear) identification $\Omega^p \to \Omega^{n-p}$ coming from the Riemannian metric. Up to some powers of $i = \sqrt{-1}$, the Hodge star operator is a grading for D.

Example

The index of D with respect to the Hodge star grading is the signature.

The Atiyah-Singer index theorem

It is not hard to prove that the index of an elliptic operator on a compact manifold M depends only on its symbol, and indeed only on the homotopy class of that symbol in the space of elliptic symbols.

The *Atiyah-Singer index theorem* gives an explicit topological formula for the index in terms of *characteristic classes* associated to the homotopy class of the symbol.

Thus it makes a connection between *topology* and *analysis*.

Example

The index theorem applied to the signature operator produces the *Hirzebruch Signature Formula*

$$\mathrm{Sign}(M) = \int_M L(M)$$

where L is a certain combination of Pontrjagin classes.

Explicitly, we have for example

Dimension of M	L-class
4	$p_1/3$
8	$(7p_2 - p_1^2)/45$
12	$(62p_3 - 13p_1p_2 + 2p_1^3)/945$

in terms of the Pontrjagin classes of the tangent bundle.

K-theory

The version of algebraic topology that is most suitable for calculations with indices is *K-theory*.

Let A be a C^*-algebra with unit. A *projection* in A is an element p such that $p = p^2 = p^*$. Let $P_n(A)$ be the collection of all projections in the matrix algebra $M_n(A)$, and let $P(A)$ be the direct limit

$$P(A) = \lim_{n\to\infty} P_n(A).$$

This is a topological space. Moreover, its set of connected components, $\pi_0(P(A))$, has a natural 'addition' operation — deform two projections until they are disjoint, then take the direct sum.

Definition

The *K-theory group* $K_0(A)$ is the Grothendieck (or universal) group associated to the abelian semigroup $\pi_0(P(A))$.

Example

The group $K_0(\mathbb{C})$ is isomorphic to $\mathbb{Z}$. The isomorphism associates to a projection in $M_n(\mathbb{C})$ the dimension of its range.

One can extend the definition of K_0 to algebras without unit. If J is an algebra without unit, embed it as an ideal in a larger algebra A. Then $K_0(J)$ is generated by formal differences $[p]-[q]$ of projections in $M_n(A)$ such that $p-q \in M_n(J)$.

Again let A be a unital algebra. A *unitary* in A is an element u such that $uu^* = u^*u = 1$. Let $U_n(A)$ denote the collection of unitaries in $M_n(A)$ and let $U(A)$ be the direct limit $\lim U_n(A)$.

Definition

The K-theory group $K_1(A)$ is the collection of connected components $\pi_0(U(A))$.

Here, because $U(A)$ is already a topological group, there is no need to use the Grothendieck construction to obtain a group.The group operation in $K_1(A)$ can be obtained from multiplication of unitaries, or from direct sum. In particular, $K_1(A)$ is abelian.

Example

The group $K_1(\mathbb{C})$ is zero, because all the unitary groups $U_n(\mathbb{C})$ are connected.

Example

Consider the algebra $A = C(S^1)$ of continuous functions on the circle. Then $K_1(A) \cong \mathbb{Z}$. The isomorphism is given by sending a loop of unitary matrices to the winding number of its determinant (a loop in $\mathbb{C} \setminus \{0\}$) about the origin.

One can also extend the definition of K_1 to non-unital algebras.

K-theory is a functor: for any $*$-homomorphism $\alpha\colon A \to B$ one can find an induced homomorphism of abelian groups

$$\alpha_*\colon K_i(A) \to K_i(B).$$

Homotopic $*$-homomorphisms induce identical maps on K-theory.

Six term exact sequence

The most important property of K-theory is the *six term exact sequence*. Let

$$0 \to J \to A \to B \to 0$$

be a short exact sequence of C^*-algebras. Then there is a commutative exact diagram

$$\begin{tikzcd}
K_0(J) \arrow[r] & K_0(A) \arrow[r] & K_0(B) \arrow[d] \\
K_1(B) \arrow[u] & K_1(A) \arrow[l] & K_1(J) \arrow[l]
\end{tikzcd}$$

The ‘connecting homomorphisms’ (vertical maps) can be described in explicit terms.

Consider the case $A = \mathfrak{B}(H)$, $J = \mathfrak{K}(H)$. The quotient $B = A/J$ is then the Calkin algebra.

Any unitary $u \in B$ lifts to $V \in A$ which is an essentially unitary Fredholm operator. On the other hand, u also defines a class $[u] \in K_1(B)$.

Theorem

The class $\partial[u] \in K_0(J) = \mathbb{Z}$ is the index of the Fredholm operator V.

Notice that the statement 'the index of an invertible operator is zero' is encoded in the exactness of the K-theory sequence.

Abstract index theory

Motivated by the above, we are going to reinterpret the index of an elliptic operator (on a compact manifold) in terms of operator K-theory. This will show us the correct generalization of the index to non-compact manifolds.

Our abstract set-up will be

- H is a Hilbert space.
- J is a C^*-algebra of operators on H.
- D is an unbounded, self-adjoint operator on H such that $f(D) \in J$ for all $f \in C_0(\mathbb{R})$.

Gradings

We distinguish the *graded* and *ungraded* cases of our basic set-up. In the ungraded case the data is exactly as I have said. In the graded case we assume in addition that

- H is a graded Hilbert space — it is equipped with a self-adjoint involution ϵ.
- The grading operator ϵ is a multiplier of J, that is $\epsilon J = J\epsilon = J$.
- D is odd relative to the grading, that is, $D\epsilon + \epsilon D = 0$.

Now let A be the *multiplier algebra* of J (the largest C^*-algebra of operators on H within which J is an ideal). There is a short exact sequence

$$0 \to J \to A \to A/J \to 0.$$

Definition

A *normalizing function* is a continuous increasing odd function $\chi\colon \mathbb{R} \to (-1,1)$ such that $\chi(t) \to \pm 1$ as $t \to \pm\infty$.

Observe

1. Any two normalizing functions differ by an element of $C_0(\mathbb{R})$.
2. The square of a normalizing function is equal to 1 modulo $C_0(\mathbb{R})$.

Thus the operator $F = \chi(D)$ is an element of $M(J)$, and its class in the quotient algebra $M(J)/J$ is a symmetry (a self-adjoint involution).

- In the *ungraded case*, a self-adjoint involution in $M(J)/J$ defines a class in $K_0(M(J)/J)$.
- In the *graded case*, an *odd* self-adjoint involution in $M(J)/J$ defines a class in $K_1(M(J)/J)$.

Definition

The *index class* of D is the image of $[F] \in K_*(M(J)/J)$ under the boundary map $\partial\colon K_*(M(J)/J) \to K_{*-1}(J)$.

Theorem

The index is an obstruction to invertibility. In other words, if 0 does not belong to the spectrum of D, then $\mathrm{Index}(D) = 0$*.*

Proof.

If the spectrum of D does not contain 0 then one can find a normalizing function χ that is equal to ± 1 everywhere on the spectrum of D. Then $\chi(D)$ itself is an involution so defines a class in $K_i(M(J))$. By exactness, the index in $K_{i-1}(J)$ is equal to zero. ☐

In the ungraded case there is nothing special about 0 here — if the spectrum of D has any gap at all, then $\mathrm{Index}(D) = 0$.

Index formulas

To conclude we give formulae for the index. These depend on knowledge of the explicit form of the connecting homomorphisms.

In the *ungraded* case the index is the homotopy class of the unitary

$$u = -e^{i\pi\chi(D)}$$

where χ is a normalizing function. (This is a unitary differing from the identity by an element of J.)
It is sometimes convenient to select χ so that u becomes the *Cayley transform*

$$(D+i)(D-i)^{-1}$$

of D.

In the *graded* case we first define a sort of ‘relative index’.

Definition

Let S, T be symmetries in $M(J)$ that anticommute modulo J (that is, $ST + TS \in J$). Define $[S : T] \in K_0(J)$ to be the K-theory class of the formal difference

$$[-STS] \ominus [T]$$

of symmetries (or of the corresponding projections).

The definition of the boundary map in K-theory gives that the index of the graded operator D is equal to $[S : \epsilon]$ where $S \in A$ is any symmetry equal to $\chi(D)$ modulo J. An example of such a symmetry is

$$S = \chi(D) + \epsilon\sqrt{1 - \chi(D)^2}.$$

We can simplify this expression.

Lemma

If $S + T$ is invertible then $[S : T] = [T : S]$.

If we choose $S = \chi(D) + \epsilon\sqrt{1 - \chi(S)^2}$ then $S + \epsilon$ is invertible so the index is equal to $[\epsilon : S]$. Now take $\chi(\lambda) = \lambda(1 + \lambda^2)^{-1/2}$; then S is the 'phase' $(D + \epsilon)|D + \epsilon|^{-1}$, and similarly $-\epsilon S \epsilon$ is the phase of $D - \epsilon$.

Thus the graded index is the difference of the phases of $D \pm \epsilon$.