Notes on Coarse Geometry

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Spring 2002

Abstract

These are the lecture notes for the Penn State course Math 597F held in Spring 2002. They will be revised and extended as the course progresses.

1 Basic definitions

In an introductory course in topology, the first step outside the world of the real numbers is usually made via the notion of a *metric space*. As you know, a metric space is a set X equipped with a distance function $d\colon X \times X \to \mathbb{R}$ which is positive, definite, symmetric, and satisfies the triangle inequality:

$$d(x_0, x_2) \leqslant d(x_0, x_1) + d(x_1, x_2).$$

It will be convenient sometimes to allow a metric to take the value ∞. One then defines the *continuity* of a function $f\colon X \to Y$ between metric spaces: f is *continuous* at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon;$$

and f is *uniformly continuous* if δ can be chosen independent of x.

When one defines continuity, one neglects a great deal of the information contained in the metric d. As every calculus student knows, only small distances matter when defining continuity. In fact, if d is a metric then so is $d' = \min\{d, 1\}$, and the topology defined by d' is exactly the same as the topology defined by d. But d' obviously erases all information about d-distances greater than one.

In this course our concern will be with a dual situation. Instead of focusing attention on the *small scale* structure defined by a metric, we will focus on the *large scale* structure. We will be able to give a precise sense to assertions such as '$\mathbb{R}$ and $\mathbb{Z}$ are the same on the large scale', or '$\mathbb{R}$ and $\mathbb{R}^2$ are different on the large scale.'

(1.1) DEFINITION: *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a map (not necessarily continuous).*

- *(a) The map f is (metrically)* proper *if the inverse image, under f, of each bounded subset of Y, is a bounded subset of X.*
- *(b) The map f is (uniformly)* bornologous *if for every $R > 0$ there is $S > 0$ such that*

$$d(x, x') < R \Rightarrow d(f(x), f(x')) < S.$$

- *(c) The map f is* coarse *if it is proper and bornologous.*

For example, if $X = Y = \mathbb{N}$, the natural numbers, then the map $n \mapsto 14n + 78$ is coarse, but the map $n \mapsto 1$ is not coarse (it fails to be proper), and the map $n \mapsto n^2$ is not coarse either (it fails to be bornologous).

One says that $f\colon X \to Y$ is *large-scale Lipschitz* if there are positive constants c and A such that

$$d(f(x), f(x')) \leqslant c d(x, x') + A.$$

Clearly a large scale Lipschitz map is bornologous.

There is a natural definition of the *length* (possibly $+\infty$) of a continuous curve in a metric space; and it follows trivially from the triangle inequality that the length of any curve from x to x' is bounded below by $d(x, x')$. One says that X is a *length space* if $d(x, x')$ is equal to the infimum of the lengths of closed curves joining x and x'. Many familiar examples of metric spaces are in fact length spaces, including all Riemannian manifolds. It is a simple exercise to prove

(1.2) LEMMA: *Let X be a length space, Y any metric space. Then the following properties of a map $f\colon X \to Y$ are equivalent:*

(a) f is large scale Lipschitz;

(b) f is bornologous;

(c) There exist $R, S > 0$ such that $d(x, x') < R \Rightarrow d(f(x), f(x')) < S$.

Two maps f, f' from X (a set) into a metric space Y are *close* if $d(f(x), f'(x))$ is bounded, uniformly in X. We say that metric spaces X and Y are *coarsely equivalent* if there exist coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and on X respectively.

It is easy to check (exercise!) that f and f' are close if and only if they arise from a coarse map $X \times [0, 1] \to Y$ by evaluating at 0 and at 1. For this reason I originally used the terminology 'bornotopic' for 'close', and 'bornotopy equivalence' for 'coarse equivalence'.

A basic example of a coarse equivalence is the inclusion $\mathbb{Z} \to \mathbb{R}$. An inverse up to closeness is provided by the map $\mathbb{R} \to \mathbb{Z}$ which assigns to each real number x the greatest integer less than or equal to x.

EXERCISE 1.3: Suppose that X and Y are metric spaces. A map $f\colon X \to Y$ is *effectively proper Lipschitz* (EPL) if there are positive constants c, A such that

$$c^{-1} d(x, x') - A \leqslant d(f(x), f(x')) \leqslant c d(x, x') + A.$$

A subset Z of Y is *coarsely dense* if there is some $R > 0$ such that every point of Y lies within R of a point of Z. Show that if $f\colon X \to Y$ is an EPL map and $f(X)$

is coarsely dense in Y, then f is a coarse equivalence. Moreover, if both X and Y are length spaces, then every coarse equivalence arises in this way. However, this conclusion does not follow if only one of the spaces X, Y is a length space.

EXERCISE 1.4: Let $X = \{2^{2^n} : n \in \mathbb{N}\}$ with the metric induced from $\mathbb{R}$. Show that every bijective map $X \to X$ is a coarse equivalence, but that such a map is EPL if and only if it is the identity outside a finite set.

1.1 Groups as coarse spaces

Let Γ be a group and $F \subseteq \Gamma$. One says that F *generates* Γ if no proper subgroup of Γ contains F. This implies that every $\gamma \in \Gamma$ can be written as a *word* in the members of F and their inverses; that is a product

$$\gamma = \gamma_1\gamma_2\cdots\gamma_n$$

where each γ_i is either a member of F or the inverse of a member of F. The smallest number n of generators that can be used is called the *word length* of γ relative to F, and is written $|\gamma|_F$.

The following lemma is obvious:

(1.5) LEMMA: *For* $\gamma, \delta \in \Gamma$ *one has* $|\gamma\delta| \leqslant |\gamma| + |\delta|$. $\square$

Consequently, the formula $d(x, y) = |x^{-1}y|$ defines a *metric* on Γ, which by construction is invariant under the left-translation[1] action of Γ on itself. We call it the *word metric* associated to the generating set F.

The word metric of course depends on the choice of generating set. However, its coarse equivalence class does not:

(1.6) PROPOSITION: *Let* F *and* F' *be two finite generating sets for the same group* Γ*, and let* d *and* d' *be the associated word metrics. Then the identity map* $(\Gamma, d) \to (\Gamma, d')$ *is a coarse equivalence.*

PROOF: Let $c = \max\{|\gamma|_F : \gamma \in F'\}$. Then the F'-word length of any group element is at most c times its F-word length, and thus

$$d'(x, x') \leqslant c d(x, x').$$

This shows that the identity map from (Γ, d) to (Γ, d') is a coarse map. Equally, the identity map from (Γ, d') to (Γ, d) is a coarse map, and the result follows. $\blacksquare$

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[1] We could also define a right-invariant metric by $d(x, y) = |xy^{-1}|$. Open question: for which groups Γ does the identity map give a coarse equivalence between the left and right invariant word metrics? Coarse modular theory?

What this tells us is that any finitely generated group carries an *intrinsic* coarse geometry. Very many structural properties of the group are determined by this coarse-geometric information.

Let us recall some terminology about group actions. Let Γ be a group acting on a topological space X. The action is *cocompact* if there is a compact $K \subseteq X$ such that $\bigcup_\gamma \gamma K = X$. It is *proper* if each $x \in X$ has a neighborhood[2] U such that $\gamma U \cap U = \emptyset$ for all but finitely many γ. To orient the mind, let M be a compact Riemannian manifold, X its universal cover, $\Gamma = \pi_1(M)$ acting on X by deck transformations.

(1.7) THEOREM: *(Švarc-Milnor) Suppose that X is a proper*[3] *length space and Γ a finitely generated group acting properly and cocompactly by isometries on X. Then Γ is coarsely equivalent to X. The equivalence is implemented by the map $\gamma \mapsto \gamma \cdot x_0$, for any fixed base point $x_0 \in X$.*

Historically this was a key result in looking at groups as geometric objects. The hypothesis that X be proper is redundant: the Hopf-Rinow theorem for length spaces implies that any such space that admits a proper, cocompact group action is itself proper. Similarly The hypothesis of finite generation of Γ is redundant; any group that acts properly and cocompactly on a length space is finitely generated.

PROOF: Fix x_0 and define $f\colon \Gamma \to X$ by $f(\gamma) = \gamma \cdot x_0$, as suggested above.

Choose $A > 0$ such that the translates of $B(x_0; A)$ cover X (such an A exists by cocompactness). Define $g\colon X \to \Gamma$ by sending $x \in X$ to any $\gamma \in \Gamma$ such that $x \in \gamma \cdot B(x_0; A)$.

We must show that f and g are coarse maps and that their composites are close to the identity.

(a) f *is coarse.* For every $R > 0$ there are only finitely many $\gamma \in \Gamma$ with $|\gamma| \leqslant R$, so we may define $S = \max\{d(x_0, \gamma \cdot x_0) : |\gamma| \leqslant R\}$. Then, since the action of Γ is by isometries,

$$d(\gamma, \gamma') \leqslant R \Rightarrow |\gamma^{-1}\gamma'| \leqslant R \Rightarrow d(x_0, \gamma^{-1}\gamma' \cdot x_0) \leqslant S \Rightarrow d(\gamma \cdot x_0, \gamma' \cdot x_0) \leqslant S,$$

so f is coarse.

(b) g *is coarse* Since X is proper, the closed ball $\overline{B}(x_0; 4A)$ is compact, and therefore there are only finitely many group elements γ for which

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[2] This agrees with the usual definition, which replaces U by a compact K, provided that X itself is a locally compact space.

[3] This means that closed bounded sets are compact.

$\gamma B(x_0;4A) \cap B(x_0;4A) \neq \emptyset$. Let k be the maximum of their word lengths. Suppose now that $x, x' \in X$ with $d(x,x') \leqslant A$ and $g(x) = \gamma$, $g(x') = \gamma'$. Then (by the triangle inequality) $d(\gamma x_0, \gamma' x_0) \leqslant 3A$ and so $\gamma B(x_0;4A) \cap \gamma' B(x_0;4A) \neq \emptyset$. We conclude that $|\gamma^{-1}\gamma'| \leqslant k$, so $d(\gamma,\gamma') \leqslant k$. We have proved that

$$d(x,x') \leqslant A \Rightarrow d(g(x), g(x') \leqslant k.$$

Because X is a length space, this suffices (Lemma 1.2) to show that g is coarse.

(c) If $\gamma' = g(f(\gamma))$, then $d(\gamma' x_0, \gamma x_0) \leqslant A$. By the same argument as above it follows that $|\gamma^{-1}\gamma'| \leqslant k$, hence $g \circ f$ is close to the identity on Γ.

(d) If $x' = f(g(x))$ then, by construction, $d(x', x) \leqslant A$. Thus $f \circ g$ is close to the identity on X.

This completes the proof. ■

(1.8) COROLLARY: *Let Γ be a finitely generated group and Γ' a subgroup of finite index. Then the inclusion $\Gamma' \to \Gamma$ is a coarse equivalence.*

PROOF: Let X be the *Cayley graph* of Γ; the graph whose vertices are the points of Γ and where two vertices are joined by an edge if the corresponding group elements are related by multiplication by a generator. It is a length space in a natural way. The Svarc-Milnor theorem shows that $\Gamma \to X$ and $\Gamma' \to X$ are both coarse equivalences. ■

Two groups Γ_1 and Γ_2 are *commensurable* if they have isomorphic finite-index subgroups. By the above corollary, commensurable groups are coarsely equivalent. The converse is false (example later? Ghys and de la Harpe p 10, using Mostow rigidity; Bridson-Haefliger for Sol geometry.)

1.2 Axiomatic coarse geometry

The basic ideas of coarse geometry do not depend strongly on a particular choice of metric, and for this reason it is helpful to axiomatize the situation in a more metric independent way. (Compare the passage from metric to topological spaces. As in that case, it will be useful to sometimes to consider 'nonmetrizable' structures.) The key notion is as follows. Let X be a metric space and let $E \subseteq X \times X$. We say that E is *controlled* if the coordinate projection maps $\pi_1, \pi_2 \colon E \to X$ are close; otherwise put, the supremum

$$\sup\{d(x, x') : (x, x') \in E\}$$

is finite. It is easy to see that the controlled sets associated to a metric space X have the following properties:

(i) The diagonal of $X \times X$ is controlled;

(ii) Any subset of a controlled set is controlled;

(iii) The transpose $E^t = \{(x', x) : (x, x') \in E\}$ of a controlled set E is controlled;

(iv) The product $E' \circ E'' = \{(x', x'') : \exists x \in X,\ (x', x) \in E',\ (x, x'') \in E''\}$ of two controlled sets E' and E'' is controlled;

(v) The union of two controlled sets is controlled.

(1.9) DEFINITION: *Let* X *be a set. A collection* $\mathcal{E}$ *of 'controlled' subsets of* $X \times X$ *that satisfies axioms (i)–(v) above is called a* coarse structure *on* X.

If our metrics are not allowed to take the value $+\infty$, then the controlled sets will have the following additional property:

(vi) Each point of $X \times X$ belongs to some controlled set.

A coarse structure having this property will be called *connected.*

(1.10) REMARK: Suppose that we are given a family $\{\mathcal{E}_\alpha\}$ of coarse structures on X. Then the intersection $\bigcap_\alpha \mathcal{E}_\alpha$ is also a coarse structure on X. In a standard way, this allows us to define the coarse structure *generated* by a collection of subsets of $X \times X$; it is the intersection of all the coarse structures containing the given subsets; there is always at least one such coarse structure, given by the collection of *all* subsets of $X \times X$.

We have seen that each metric on X defines a coarse structure. But there are also important examples of coarse structures that do not come from any metric. To fix our attention on a specific example, let X be Euclidean space $\mathbb{R}^2$. It is natural to regard X as admitting a circle of 'ideal points' at infinity, and there is a well-defined notion of what it means for a sequence in X to converge to such an ideal point. (Formally speaking what we are doing here is identifying X with the interior of the unit disc by a radial homeomorphism.) Let us define a coarse structure on X by the following prescription: a subset E of $X \times X$ is controlled if and only if, whenever (x_n, x'_n) is a sequence in E and one of the sequences $\{x_n\}$, $\{x'_n\}$ converges to an ideal point, then the other one also converges to the same ideal point.

EXERCISE 1.11: Check that this does indeed give a coarse structure on X according to our definition.

Such a coarse structure can be defined for any locally compact metrizable space X equipped with a compactification $\overline{X}$. It is sometimes called a *continuously controlled* coarse structure. These structures are non-metrizable. It suffices to show that the continuously controlled structure on $\mathbb{N}$, one-point compactified by a point at ∞, is non-metrizable. To see this, suppose that d were a metric on the natural numbers defining the continuously controlled coarse structure. Then we would have

(a) d is unbounded;

(b) For any two sequences x_n, x'_n tending to infinity, $d(x_n, x'_n)$ is bounded.

For any n, the unboundedness of the metric d gives us a natural number $x_n > n$ with $d(x_n, n) \geqslant n$. This contradicts (b), since both $\{x_n\}$ and $\{n\}$ tend to infinity.

It is natural to ask whether there is a 'metrization theorem' for coarse structures. Such a result can be found.

(1.12) THEOREM: *A coarse structure on X is metrizable if and only if it is countably generated.*

PROOF: If $\mathcal{E}$ is given by the metric d, then it is generated by the sets $E_n = \{(x, x') : d(x, x') \leqslant n\}$ for $n = 1, 2, \ldots$. Suppose conversely that $\mathcal{E}$ is generated by a countable family of sets E_n, $n = 1, 2, \ldots$. Inductively define F_0 to be the diagonal and

$$F_n = (F_{n-1} \circ F_{n-1}) \cup E_n \cup E_n^t.$$

Then the F_n are symmetric ($F_n = F_n^t$) and satisfy $\Delta \subseteq F_{n-1} \subseteq F_{n-1} \circ F_{n-1} \subseteq F_n$. It follows that the collection of all subsets of $X \times X$ that are contained in some F_n is a coarse structure $\mathcal{F}$. Since each generator E_n belongs to $\mathcal{F}$, we have $\mathcal{E} \subseteq \mathcal{F}$; on the other hand, each F_n belongs to $\mathcal{E}$ and so $\mathcal{F} \subseteq \mathcal{E}$. Hence $\mathcal{E} = \mathcal{F}$ and the F_n generate $\mathcal{E}$.

Now set

$$d(x,x') = \inf\{n : (x,x') \in F_n\}.$$

I claim that d is a metric. We check the triangle inequality: if $(x,x') \in F_n$ and $(x',x'') \in F_m$, where without loss of generality $n \geqslant m$, then $(x,x'') \in F_n \circ F_n \subseteq F_{n+1}$. This gives us the triangle inequality in the strong form

$$d(x,x'') \leqslant \max\{d(x,x'), d(x',x'')\} + 1$$

for $x \neq x' \neq x''$.

Notice that d may take the value $+\infty$; it does so if and only if the coarse structure $\mathcal{E}$ is disconnected. ■

(1.13) DEFINITION: *Let X be a coarse space and S a set. Two maps $f, f' \colon S \to X$ are* close *if the set $\{(f(s), f'(s)) : s \in S\} \subseteq X \times X$ is controlled.*

Closeness is an equivalence relation. It is a tautology that $E \subseteq X \times X$ is controlled if and only if the two coordinate projections are close on E.

(1.14) PROPOSITION: *Let X be a coarse space, and let B be a subset of X. The following are equivalent:*

(a) $B \times B$ is controlled;

(b) $B \times \{p\}$ is controlled for some $p \in X$;

(c) The inclusion map $B \to X$ is close to a constant map.

A set B satisfying these conditions is called *bounded.*

PROOF: Without loss of generality $B \neq \emptyset$. Pick $p \in B$. Then $(a) \Rightarrow (b)$ because $B \times \{p\} \subseteq B \times B$. On the other hand $(b) \Rightarrow (a)$ because

$$B \times B = (B \times \{p\}) \circ (\{p\} \times B).$$

Finally, (c) is simply a restatement of (b). ■

In a metric coarse structure, the bounded sets are just the metrically bounded ones.

(1.15) LEMMA: *In a connected coarse structure, the union of two bounded sets is bounded.*

The proof is obvious.

Exercise 1.16: Show that if X is a locally compact space equipped with a continuously controlled structure coming from a metrizable compactification (as in the example of $\mathbb{R}^2$ above), then the bounded subsets of X are precisely those that have compact closure.

Now that we have the notions of closeness, control, and bounded set, we can define the basic notions of coarse geometry exactly as we did for metric spaces. Thus

(1.17) Definition: *Let X and Y be coarse spaces, and let $f\colon X \to Y$ be a map.*

- *(a) The map f is* proper *if the inverse image, under f, of each bounded subset of Y, is a bounded subset of X.*
- *(b) The map f is* bornologous *if for controlled subset E of $X \times X$, the set $(f \times f)(E)$ is a controlled subset of $Y \times Y$.*
- *(c) The map f is* coarse *if it is proper and bornologous.*
- *(d) The spaces X and Y are* coarsely equivalent *if there exist coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and on X respectively.*

Exercise 1.18: Suppose X and Y have continuously controlled coarse structures and that $f\colon X \to Y$ is a continuous and proper map. When is it a coarse map?

If X is a topological space, it is natural to require some compatibility between a coarse structure on X and its topology. We say that a coarse structure on X is *proper* if

- (i) There is a controlled neighborhood of the diagonal, and
- (ii) Every bounded subset of X has compact closure.

Since there is a controlled neighborhood of the diagonal, every point of X has a bounded neighborhood; so every compact subset of X is bounded. Thus the bounded subsets of X are *exactly* those that have compact closure. A proper metric space is therefore a proper coarse space.

Exercise 1.19: Continuously controlled structures are proper.

Exercise 1.20: C_0 structures.

1.3 Hyperbolic space

In this section we will take a look at the most important single example of a coarse geometric structure.

Let X be a metric space. A ball in X is defined by its center and radius. Let $\mathcal{H}(X)$ be the space of balls in X, so that $\mathcal{H}(X) = X \times \mathbb{R}^+$. We want to provide $\mathcal{H}(X)$ with a coarse structure. What should it mean to say that two balls in X are close?

A moments thought suggests that two balls should be thought of as close if their intersection comprises a large proportion of each one of them. For this to be possible, both balls must be of more or less the same size, and their centers must be close relative to their radii. To be precise, let us define a subset $E \subseteq \mathcal{H} \times \mathcal{H}$ as follows: $((x,r),(x',r'))$ belongs to E if $\frac{1}{2} \leqslant r/r' \leqslant 2$ and $d(x,x') \leqslant \frac{1}{2}\min\{r,r'\}$.

This single set E generates a coarse structure on $\mathcal{H}(X)$, which is connected provided X is connected[4].

(1.21) DEFINITION: *We will call* $\mathcal{H}(X)$ *the* hyperbolic space *built on* X. *The* hyperbolic plane $\mathbb{H}^2$ *is the hyperbolic space built on* $\mathbb{R}$.

The coarse structure of $\mathcal{H}(X)$ is metrizable, by Theorem 1.12. We will find an explicit formula for the metric giving the coarse structure on $\mathbb{H}^2$.

(1.22) PROPOSITION: *The Riemannian metric on* $\mathbb{H}^2$ *given by*

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{|\Im z|^2}$$

(where we identify $\mathbb{H}^2 = \{z = x + iy : y > 0\} \subseteq \mathbb{C}$*) gives rise to the hyperbolic coarse structure.*

PROOF: Let d be the distance function associated to the above Riemannian metric. Let E be the generating set for the hyperbolic coarse structure described above. Then it is necessary to show that there are two constants $\alpha, \beta > 0$ such that if $d(z_0,z_1) < \alpha$ then $(z_0,z_1) \in E$ and if $(z_0,z_1) \in E$ then $d(z_0,z_1) < \beta$.

Let $z_0 = x_0 + iy_0$, $z_1 = x_1 + iy_1$ and suppose that $y_1 \leqslant y_0$. Let $z' = x_0 + iy_1$. Then (by integrating along Euclidean straight line paths), if $(z_0,z_1) \in E$ then $d(z_0,z') \leqslant \log 2$ and $d(z',z_1) \leqslant \frac{1}{2}$, so we may take $\beta = \frac{1}{2} + \log 2$.

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[4] In the coarse sense: the metric does not take the value $+\infty$.

Set $\alpha = \frac{1}{4}$. If $d(z_0, z_1) < \alpha$ then there is a path $\gamma(t) = z_t$ of length less than $\frac{1}{3}$ joining z_0 and z_1. The length of γ is given by the integral

$$\int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} \geqslant \int_\gamma \frac{|dy|}{y}.$$

It follows that the y-coordinates y_t all satisfy $\frac{2}{3}y_0 \leqslant y_t \leqslant \frac{3}{2}y_0$ (here we use $e^{1/3} < \frac{3}{2}$). Moreover

$$|x_0 - x_1| \leqslant \int_\gamma |dx| \leqslant \frac{3y_0}{2} \int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} \leqslant \frac{1}{2}.$$

Thus $(z_0, z_1) \in E$, as required. ■

The Riemannian metric $(dx^2 + dy^2)/y^2$ on the upper half plane is a very classical object.

(1.23) Definition: *A* geodesic *in a metric space* X *is an isometric embedding of an interval of* $\mathbb{R}$.

Note that this is a stronger notion than the standard one in Riemannian geometry — sometimes it's called a *global geodesic*. In particular, any geodesic automatically minimizes the distance between its endpoints. A length space is called a *geodesic space* if any two points can be joined by a geodesic.

(1.24) Lemma: *The vertical Euclidean straight lines in* $\mathbb{H}^2$ *are geodesics.*

Proof: Easy estimates as in the proof above show that the hyperbolic distance from $x + iy$ to $x + iy'$ is at least $|\log y/y'|$. But this distance is realized by the Euclidean line segment between these points. ■

2 Limits

Our basic idea in this section is to study the coarse geometry of a metric space (X, d) by the following procedure. Denote by X_j the rescaled space $(X, \frac{1}{j}d)$ in which all distances are divided by j. If the 'limit space' $X_\infty = \lim_{j\to\infty} X_j$ exists in some well-defined sense, then its ordinary geometry should capture (some of) the large scale geometry of X.

To make this precise, we begin by studying the convergence of metric spaces.

2.1 Gromov-Hausdorff convergence

Let X be a fixed metric space and let $A, B \subseteq X$. The *Hausdorff distance* in X between A and B is, by definition,

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N_\varepsilon(B),\ B \subseteq N_\varepsilon(A)\}$$

where $N_\varepsilon(A)$ means the union of the open ε-balls around all points of A. It is easy to see that d_H makes the collection of *closed* subsets of X into a metric space (closedness is needed in order that $d_H(A, B) = 0$ should imply $A = B$).

(2.1) PROPOSITION: *Let X be a compact metric space. Then the collection $\mathcal{C}(X)$ of closed subsets of X (equipped with the Hausdorff metric) is also a compact metric space.*

PROOF: Let C_i be a sequence of closed subsets of X. We'll show that it has a Hausdorff-convergent subsequence.

Observe first that $\{C_i\}$ is *uniformly precompact*: given any $\varepsilon > 0$ there is N_ε such that each space C_i can be covered by at most N_ε balls of radius ε. To see this, first cover X itself by a finite number N_ε of balls of radius $\varepsilon/2$, and then observe that each such that meets a given closed set C is contained in a ball of radius ε centered on a point of C.

We can therefore construct in each C_i a sequence $\{x(i,j)\}_{j=1}^\infty$ in such a way that these sequences are 'uniformly dense' in C_i: for each $\varepsilon > 0$ there is J_ε such that, for each i, the union of the ε-balls around $x(i,1), \ldots, x(i, J_\varepsilon)$ is equal to the whole of C_i.

For fixed j, each sequence $\{x(i,j)\}_{i=1}^\infty$ has a convergent subsequence. Using a diagonal procedure, we may extract a subsequence of the C_i's along which *each* $x(i,j)$ converges (as $i \to \infty$ with j fixed) to a point $x(\infty, j)$. For simplicity of notation, assume that $\{C_i\}$ itself has this property (in other words, assume that we have already passed to a suitable subsequence).

Let C be the closure (in X) of the set $\{x(\infty, j) : j = 1, 2, \dots\}$. A '$3\varepsilon$-argument' now shows that C_j converges to C in the Hausdorff metric. ■

In the absence of an ambient space such as the X above, Gromov defined the distance between A and B as an infimum over all possible ambient spaces.

(2.2) DEFINITION: *Let A and B be compact metric spaces. The* Gromov-Hausdorff distance *between them is defined by*

$$d_{GH}(A, B) = \inf d_H(A, B)$$

where the distance on the right is the Hausdorff distance between A and B in some compact metric space Z containing isometrically embedded copies of A and B, and the infimum is taken over all such metric spaces Z.

(2.3) REMARK: Such metric spaces Z always exist — consider a metric on the disjoint union $A \sqcup B$ with large distance (greater than the sum of the diameters of A and B) between every point of A and every point of B. One obtains the same definition if, instead of taking the infimum over *all* Z, one merely takes the infimum over metrics on $Z = A \sqcup B$ that restrict to the given metric on A and on B.

(2.4) PROPOSITION: *d_{GH} is a metric on the collection of isometry classes of proper metric spaces.*

PROOF: The only point that deserves a little discussion is the implication $d_{GH}(A, B) = 0$ implies A, B isometric. Choose a dense sequence $\{a_i\}_{i=1}^{\infty}$ in A. By definition, for each j one can find a metric d_j on $A \sqcup B$ for which the Hausdorff distance between A and B is less than $1/j$, and for each i one may then select $b_{i,j} \in B$ such that the distance $d_j(a_i, b_{i,j})$ is less than $1/j$. Using a diagonal process, we may pass to a subsequence of the j's for which $b_{i,j}$ converges, for each fixed i, to $b_i \in B$. The map $a_i \mapsto b_i$ is isometric, and so (by density) extends uniquely to an isometry from A into B. ■

Bringing out an idea which appeared in an earlier proof, let us say that a collection of compact metric spaces is *uniformly compact* if their diameters are uniformly bounded and, given any $\varepsilon > 0$, there is a number N_ε such that each space in the collection can be covered by at most N_ε balls of radius ε.

(2.5) THEOREM: (GROMOV'S COMPACTNESS CRITERION) *If* $\{C_i\}_{i=1}^{\infty}$ *is a uniformly compact sequence of compact metric spaces, then it has a Gromov-Hausdorff convergent subsequence.*

We will need the following lemma, whose proof is an exercise.

(2.6) LEMMA: *Let* X *be any metric space, let* S *be a dense subset of* X. *The mapping* $X \to \ell^{\infty}(S)$ *that sends* $x \in X$ *to the function* $s \mapsto d(x,s)$ *is an isometric embedding.*

PROOF: Observe that it will be enough to embed the C_i isometrically into some fixed compact metric space Z. Then we can apply Proposition 2.1.

Let N_k be the number of $1/k$-balls necessary to cover any one of the sets C_i. Let A_k be the finite set

$$\{1,\ldots,N_1\} \times \{1,\ldots,N_2\} \times \cdots \times \{1,\ldots,N_k\}$$

with $p_k \colon A_k \to A_{k-1}$ the natural projection map; let A be the disjoint union of the A_k. In each of the C_i choose (by induction on k) points $x_{i,a}$ for each $a \in A$ such that

(a) the balls of radius 1 about the points $x_{i,a}$, $a \in A_1$, cover X_i

(b) each $x_{i,a}$ where $a \in A_k$, $k > 1$, lies in the ball of radius $2/(k-1)$ centered at $x_{i,a'}$, where $a' = p_k(a)$; and for each $a' \in A_{k-1}$ the balls of radius $1/k$ around all $x_{i,a}$ with $p_k(a) = a'$ cover the ball of radius $1/(k-1)$ around a'.

Consider now the maps $h_i \colon C_i \to \ell^{\infty}(A)$ defined by

$$h_i(x) : a \mapsto d(x, x_{i,a}).$$

By the lemma above, these are isometric embeddings. Moreover the $h_i(x) = \varphi$ all satisfy the following inequalities:

(a) $0 \leqslant \varphi(a) \leqslant d$ for all a (d is the upper bound for the diameter provided by uniform compactness), and

(b) $|\varphi(a) - \varphi(p_k(a))| \leqslant 2/(k-1)$ for all $k > 1$.

The space of all functions φ satisfying these two criteria is a compact metric space (it is a product of Euclidean cubes with the product topology). Thus we have isometrically embedded all the C_i in the same compact space, so we are done. ■

(2.7) Corollary: *The Gromov-Hausdorff space of compact metric spaces is complete.*

Proof: It is easy to see that a sequence of compact metric spaces that is Cauchy in the Gromov-Hausdorff metric is uniformly compact. ■

We will be interested mostly in the convergence of sequences of metric spaces which are not compact but are *proper* (recall that this means that closed bounded sets are compact). We consider *pointed* proper metric spaces; that is, proper metric spaces X provided with a choice of basepoint $x \in X$. If (X_n, x_n) is a sequence of pointed proper metric spaces we say that it *converges* to (X, x) if, for all $R > 0$, the closed ball $\overline{B}_{X_n}(x_m; R)$ converges in the Gromov-Hausdorff metric to the closed ball $\overline{B}_X(x; R)$.

Example: The d-spheres S^d_n of radius n, with the north pole as basepoint, converge as $n \to \infty$ to Euclidean space $\mathbb{R}^d$.

Let X be a fixed metric space and let αX denote the metric space whose underlying set is the same but whose distances are rescaled by a factor α. Let's investigate some examples of the phenomenon mentioned in the introduction, whether the spaces $\frac{1}{j}X$ converge to something as $j \to \infty$.

Example: Let $X = \mathbb{Z}^d$ with the word metric coming from the usual generators. It is then almost obvious that $\frac{1}{j}X$ converges to $X_\infty = \mathbb{R}^d$ equipped with the ℓ^1 metric. Indeed, there are isometric embeddings $\frac{1}{j}X \to \mathbb{R}^d$ whose images are $\frac{1}{j}$-dense. Notice that X is at finite (GH) distance from X_∞ in this case; the refinements of (proper, pointed) convergence are not required.

We can generalize this example. Let $X = \mathbb{Z}^d$ equipped with any translation invariant metric at all (any word metric is okay, in particular). We claim that $\frac{1}{j}X$ converges to $\mathbb{R}^d$ equipped with some norm. To prove this, notice first that the metric is determined by the function $d = d(\cdot, e)$ on the group. Embed $\mathbb{Z}^d$ into $\mathbb{R}^d$ and extend the function d to $\mathbb{R}^d$ by choosing a fundamental domain F for the action of $\mathbb{Z}^d$ on $\mathbb{R}^d$ and defining $d(v) = d(\gamma)$ where $\gamma \in \mathbb{Z}^d$ has $v \in \gamma + F$. Notice that the extended function d satisfies the 'approximate triangle inequality'

$$d(u+v) \leqslant d(u) + d(v) + C,$$

where C is an absolute constant.

(2.8) Lemma: *In the situation above, the limit*

$$\|v\| = \lim_{k\to\infty} (1/k)d(kv)$$

exists for any vector $v \in \mathbb{R}^d$ *and defines a norm.*

PROOF: Fix v and define $a_k = d(kv) - (k-1)C$. Then we easily find

$$a_{k+k'} \leqslant a_k + a_{k'}.$$

A well known calculus lemma states that if $\{a_k\}$ is a sequence of real numbers having this property then a_k/k converges to a limit (possibly $-\infty$) as $k \to \infty$. Applying this in the present case we find that

$$\frac{1}{k}d(kv) = \frac{a_k}{k} + \frac{k-1}{k}C$$

converges to a limit, necessarily $\geqslant 0$, which we denote $\|v\|$. The triangle inequality

$$\|u+v\| \leqslant \|u\| + \|v\|$$

is immediate from the approximate triangle inequality for d. The scaling property

$$\|\lambda u\| = \lambda\|u\|, \quad (\lambda \geqslant 0)$$

is true by definition for $\lambda \in \mathbb{Z}$, and hence for $\lambda \in \mathbb{Q}$. The general case follows by continuity: $\|\cdot\|$ is continuous since d has an 'approximate continuity' property, there exist ε and δ such that $|u-v| < \delta$ implies $|d(u) - d(v)| < \varepsilon$. ■

We have now produced in a natural way a norm on $\mathbb{R}^d$ which arises as the 'rescaled limit' of the metrics on $\mathbb{Z}^d$. 'Obviously' this should be the Gromov-Hausdorff limit of the spaces $\frac{1}{j}\mathbb{Z}^d$ but some more work is needed to prove this. (It is obvious that $\mathbb{R}^d$ with this norm is the asymptotic cone — see later.) What we need is a result from Dima Burago's thesis:

(2.9) PROPOSITION: *Let* d *be the distance-from-the-origin function on* $\mathbb{Z}^d$*, as above, coming from a metric on* $\mathbb{Z}^d$ *which is either a word metric, or is induced by a* $\mathbb{Z}^d$*-periodic Riemannian metric on* $\mathbb{R}^d$*. Then there is some absolute constant* C *such that*

$$d(2x) \geqslant 2d(x) - C$$

for all $x \in \mathbb{Z}^d$.

It follows immediately that $d(2^k x) \geqslant 2^k d(x) - (2^k - 1)$. Therefore $\mathbb{Z}^d$ is at finite Hausdorff distance from $(\mathbb{R}^d, \|\cdot\|)$ and the rescaled spaces $\frac{1}{j}\mathbb{Z}^d$ Hausdorff-converge to $\mathbb{R}^d$.

PROOF: This is easy for a word metric. If $2x = a_1x_1 + \cdots + a_mx_m$ is the shortest expression of $2x$ as a word in the generators $x_1, \ldots, x_m$ (where $m \geqslant d$), so that $d(2x) = |a_1| + \cdots + |a_m|$, then consider the element $y = b_1x_1 + \cdots + b_mx_m$, where b_j is the integer part of $a_j/2$. This represents an element of $\mathbb{Z}^d$ which is at bounded Euclidean distance from x (at most the sum of the Euclidean norms of the generators) and therefore $d(y-x)$ is bounded. Since $d(y) \leqslant \frac{1}{2}d(2x) - m$, the result follows.

The argument for a metric coming from $\mathbb{R}^d$ is much more subtle. The key geometric lemma is the following

(2.10) LEMMA: *Let γ be a continuous path in $\mathbb{R}^d$ joining the origin to a vector v. Then γ can be divided into at most $d+1$ subintervals, some of which can be rearranged by translation*[5] *to form a continuous path from the origin to $v/2$.*

To apply this lemma to prove the proposition, (explain later — with figure).

The lemma is proved by algebraic topology. Assume γ is parameterized by $[0,1]$. Let $\mathbf{t} = (t_0, \ldots, t_d)$ lie on the set $\sum t_i^2 = 1$, which is a topological d-sphere. Partition $[0,1]$ correspondingly into $d+1$ subintervals $I_0, \ldots, I_d$ given by

$$0 \leqslant t_0^2 \leqslant t_0^2 + t_1^2 + \leqslant \cdots \leqslant 1$$

and count I_j as 'positive' if the corresponding t_j is $\geqslant 0$, 'negative' otherwise. The function

$$\mathbf{t} \mapsto \int_+ \gamma'(t)dt - \int_- \gamma'(t)dt,$$

where $\int_+$ denotes integration over the positive I_j and $\int_-$ denotes integration over the negative I_j, is then an *antipodal* map from S^d to $\mathbb{R}^d$. Thus, according to the Borsuk-Ulam Theorem, it takes the value zero somewhere, as required. ■

EXAMPLE: Let $X = \mathbb{F}_2$, the (underlying metric space of the) *free group* on 2 generators. (There is nothing special about 2 here. Indeed, exercise, all finitely generated nonabelian free groups are coarsely equivalent.) We will show that the rescaled space $(1/j)X$ do not converge in the pointed Gromov-Hausdorff topology; in fact, the sequence $(1/j)B(e;j)$ is not uniformly precompact. If it were, then for fixed $\varepsilon > 0$ we could cover $\frac{1}{j}B(e;j)$ by $N(\varepsilon)$ balls of radius ε; thus we could cover $B(e;j)$ itself by $N(\varepsilon)$ balls of radius $j\varepsilon$. Taking, say, $\varepsilon\frac{1}{2}$ we get

$$\#B(e;j) \leqslant N(\varepsilon)\#B(e;j/2);$$

(all balls of the same radius have the same number of elements.) But we know that $\#B(e;r) = 1 + 4 \cdot 3^{r-1}$, and taking r sufficiently large we get a contradiction. (Of course, the same argument applies to any space of exponential growth, such as $\mathbb{H}^2$.)

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[5] i.e. joining the head of each chosen subinterval to the tail of the next one.

In the above discussion we introduced implicitly the notion of the *growth* of a group (or of a space). We'll stick to groups for a moment. The *growth function* of a group with a word metric is the function which associates to $r > 0$ the number of elements in a ball of radius r. It depends on the metric chosen (of course), but not seriously. In particular the notions of *polynomial growth*, *exponential growth*, etc, do not depend on the metric.

The argument above will show that if a group Γ is to have the $(1/j)\Gamma$ converge to some limit space, we will need to assume that Γ has something like polynomial growth. (Remark: A metric space X is said to have *finite Assouad dimension* if there is a constant L such that, for each $r > 0$, each ball of radius $2r$ can be covered by L balls of radius r. Our arguments above show that $1/jX$ convergent implies finite Assouad dimension, which in turn implies (for groups) polynomial growth of degree about $\log L$. More on Assouad dimension later.) Conversely we have

(2.11) THEOREM: *If Γ is a group of polynomial growth, then the balls $(1/j)B(e;j)$ in Γ are uniformly precompact, and thus there is a subsequence of the $(1/j)\Gamma$ that converges in the Gromov-Hausdorff topology.*

This theorem was proved by Gromov (in *Groups of polynomial growth and expanding maps*) as the first stage in his famous program to prove that every group of polynomial growth has a nilpotent subgroup of finite index. Later, Pansu proved that in fact the spaces $(1/j)X$ *themselves* converge — there is no need to pass to a subsequence.

PROOF: The basic idea of the proof is simple enough. What we need to show is that given $A > 0$, for sufficiently large r every ball of radius Ar can be covered by some number K of balls of radius $A^{-1}r$; K, of course, should be bounded independent of r. Let $d(\rho)$ be the growth function (the number of elements in a ball of radius ρ). Imagine packing as many disjoint $A^{-1}r/2$-balls as possible inside a ball of radius Ar. The number of such balls can be at most $d(Ar)/d(A^{-1}r/2)$. On the other hand, the balls with the same centers and radius $A^{-1}r$ must cover the given Ar-ball (else there would be room to pack another one in.) It follows that we may take $K = d(Ar)/d(A^{-1}r/2)$. If d were an actual polynomial, this would certainly be bounded independent of r. In general, the growth may be more 'irregular' than an actual polynomial. The burden of Gromov's argument is to show that we can always take a subsequence of r's sufficiently 'regular' to make the above argument work. Notice that we will need to use the fact that Γ is a *group* — one can construct functions d, bounded by a polynomial, such that $d(Ar)/d(A^{-1}r/2)$ is not bounded independent of r.

Gromov defines a number r to be i-*regular* if we have inequalities for $j =$

$1, \ldots, i$:

(a) $d(2^{-j}r) \geqslant A_j d(r)$;

(b) $d(2^j r) \leqslant B_j d(r)$;

for some fixed constants A_j, B_j which depend only on the growth of the group (i.e. they don't depend on r or i). In fact, Gromov takes $A_j = 2^{-C+1j}$, where C is the exponent of polynomial growth, and $B_j = 2^{16^{j+1}(C+1)}$. It is clear then that it will suffice to produce a sequence $r_i \to \infty$ such that each r_i is i-regular. We can certainly produce a sequence of r's satisfying (a): start with the sequence 2^i, fix $j = 1$, and note that if there is no subsequence satisfying (b), then for all but finitely many r_i we have $d(r) \geqslant 2^{C+1} d(r/2)$, which implies that the growth rate is faster than C; repeat this argument for higher j's and use a diagonal procedure. So the issue is to show that for a *group* the growth is sufficiently regular that (a) implies (b).

The key is the following 'convexity' inequality for the growth function of a group:

$$d(5r)d(r) \leqslant d(4r)^2.$$

To prove the inequality, find a maximal $2r+1$-net in the ball of radius $3r$. The number of members of such a net is at most $d(4r)/d(r)$. The $2r$-balls around these centers cover the ball of radius $3r$ and therefore (by the 'connectedness at scale 1' of the group) the $4r$-balls with the same centers cover the ball of radius $5r$. This gives the inequality $d(5r) \leqslant (d(4r)/d(r))d(4r)$ as required.

We can rewrite the inequality as

$$\ell(5r) \leqslant 2\ell(4r) - \ell(r)$$

where $\ell(r) = \log d(r)$. Iterating this inequality twice

$$\ell(8r) \leqslant 16\ell(4r) - 15\ell(r)$$

(we assume that r is divisible by a high power of 2) and so

$$L(2r) \leqslant 16L(r)$$

where $L(r) = \ell(r) - \ell(r/4)$. Now iterate this last inequality j times and obtain

$$\ell(2^j r) \leqslant 16^j(\ell(r) - \ell(r/4)) + \ell(r/4) \leqslant 16^{j+1}C + \ell(r)$$

(using (a) for $j = 2$), as required. ■

2.2 Ultralimits and Asymptotic Cones

2.2.1 Ultrafilters

One might try to 'improve' the constructions of the preceding sections. There are two ways in which one could consider making such an 'improvement':

(i) One could try to define a *weaker* notion of convergence in which any set of metric spaces of uniformly bounded diameter would be 'precompact'. Notice that it is easy to give examples of spaces with bounded diameter which are not uniformly precompact, e.g. the discrete spaces with n elements, $n = 1, 2, \ldots$.

(ii) One could aim to avoid the messy process of passing to subsequences, sub-subsequences, ... by making some once and for all choice in advance.

These tasks are in principle independent, but the theory of 'ultralimits' of metric spaces accomplishes them both. There is a connection with logic here — see the paper of van den Dries and Wilkie.

Let's begin with a review of the theory of ultrafilters on a set. We'll take a slightly non-standard approach, making use of our knowledge of functional analysis. Let X be a set. The *Stone-Cech compactification of* X is the maximal ideal space of the unital commutative C^*-algebra $\ell^\infty(X)$. In other words, it is a compact Hausdorff space βX, containing X as a dense open subset, such that one can naturally identify $\ell^\infty(X) = C(\beta X)$, and such that the inclusion $X \subseteq \beta X$ induces the inclusion of the essential ideal $C_0(X)$ into $C(\beta X)$.

Here are some ways of recognizing aspects of the Stone-Cech compactification in terms of objects internal to X:

- Clopen subsets of βX are easiest to recognize. They correspond to projections in the C^*-algebra $C(\beta X)$, and thus to *arbitrary* subsets of X. (Formally speaking, given any subset $Y \subseteq X$, its closure $\overline{Y}$ in βX is a clopen subset of βX, and every clopen subset of βX arises in this way from a unique Y.)
- Every open subset U of βX has the property that $U \cap X$ is dense in U (this is a general fact about compactifications). Suppose now that

$p \in U$. By regularity there is an open V containing p with $\overline{V} \subseteq U$. Therefore (applying the previous remark to V),

$$p \in V \subseteq \overline{V \cap X} \subseteq \overline{V} \subseteq U.$$

Since $\overline{V \cap X}$ is clopen, we have shown that the clopen sets form a basis for the topology of βX (i.e. βX is zero-dimensional).

- Closed nonempty subsets of βX correspond to *filters* of subsets of X. By definition, a filter $\mathcal{F} \subseteq \mathcal{P}(X)$ has the properties that any intersection of members of $\mathcal{F}$ is a member of $\mathcal{F}$, any subset of X containing a member of $\mathcal{F}$ is also a member of $\mathcal{F}$, and $\mathcal{F}$ does not contain the empty set. To see the connection notice that given any closed subset K of βX, the collection of all F such that $\overline{F} \supseteq K$ is a filter of subsets of X. Conversely, given a filter $\mathcal{F}$ of subsets of X, the intersection $A = \bigcap_{F \in \mathcal{F}} \overline{F}$ is a closed subset of βX. These processes are inverse to one another. In one direction, this is an immediate consequence of the fact that clopen sets form a basis for the topology. In the other, we are given a filter $\mathcal{F}$ of subsets of X and we require to show that any subset S of X such that $\overline{S}$ contains the intersection $A = \bigcap_{F \in \mathcal{F}} \overline{F}$ must in fact belong to $\mathcal{F}$. Since $\overline{S}$ is open and contains the intersection $\bigcap \overline{F}$ of closed sets, it contains a finite intersection of such sets, say $\overline{F}_1, \ldots, \overline{F}_m$. Hence $S \supseteq \bigcap_{i=1}^{n} F_i$ and the result follows since $\mathcal{F}$ is a filter.

- A *point* of βX can be characterized as a *minimal* (nonempty) closed subset of X. Since the correspondence between closed subsets of βX and filters of subsets of X is inclusion-reversing, it follows that a point of βX corresponds to a *maximal* filter of subsets of X, that is an *ultrafilter*. The existence of ultrafilters can be proved by a direct Zorn's Lemma argument.

(2.12) LEMMA: *A filter $\mathcal{F}$ of subsets of X is an ultrafilter iff, for every subset S of X, either S or $X \setminus S$ (but not both, of course!) belongs to $\mathcal{F}$.*

PROOF: If $\mathcal{F}$ is a filter, and S is a set such that neither S nor $X \setminus S$ belongs to $\mathcal{F}$, then

$$\mathcal{F}' = \mathcal{F} \cup \{S \cap F : F \in \mathcal{F}\}$$

is a filter with $\mathcal{F}' \supset \mathcal{F}$. Thus maximality of $\mathcal{F}$ implies that no such set can exist. The converse is even easier. ■

By virtue of this lemma one can think of a point $\omega \in \beta X$ as defining a *finitely additive {0,1}-valued probability measure* on X: the sets of measure 1 are precisely those belonging to the filter $\mathcal{F}$, that is, those whose closures contain ω. (We'll call such sets ω-*thick*.) Note the counter-intuitive consequence that, given $\omega \in \beta X$ and any two disjoint subsets $S_1, S_2 \subseteq X$, at most one of their closures $\overline{S}_1$, $\overline{S}_2$ can contain p.

Now let ω be a point of βX (equivalently, an ultrafilter on X). If f is any bounded function on X, then f defines an element of $\ell^\infty(X) = C(\beta X)$ and thus f has a well-defined value ℓ at $\omega \in \beta X$. This value is called the ω-*limit* of f and written $\ell = \lim_\omega f(x)$; it is characterized by the fact that, for every $\varepsilon > 0$, the set

$$\{x \in X : |f(x) - \ell| < \varepsilon\}$$

is ω-thick. Of course, among the points of βX are those of X; they correspond to the so-called *principal ultrafilters*, those which contain a singleton (and therefore comprise precisely all sets containing that singleton). We're interested in the *nonprincipal ultrafilters* which correspond to points of βX. Every finite set is thin for a nonprincipal ultrafilter. It follows that if f tends to a limit at infinity (rel. the discrete topology on X) then the ultralimit always equals the regular limit. This is particularly relevant on $\mathbb{N}$.

(2.13) REMARK: *Relationship to nets:* Functional analysts may be more familiar with another definition of generalized convergence, than in terms of nets. By definition, a *net* in X is a map from a directed set to X. Each net $\{x_\alpha\}$ in X gives rise to a filter, namely the filter of those subsets $S \subseteq X$ such that $x_\alpha \in S$ for all $\alpha \geqslant \beta$ (some β; one says that $\{x_\alpha$ is 'eventually contained' in S). Ultrafilters correspond to 'universal nets', those for which, given any S, $\{x_\alpha\}$ either is eventually contained in S or is eventually contained in $X \setminus S$. A Zorn's lemma argument shows that every net has a universal subnet, just as every filter is contained in an ultrafilter. The limit of a function f along a universal net is the same as the limit that we have defined along the corresponding ultrafilter.

2.2.2 Local Convergence of Metric Spaces

Many interesting properties of metric spaces X depend on the distance properties of finite sets of points in X. For instance, the triangle inequality is a 'three point property'. The parallelogram law (the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of all four sides) characterizes Euclidean space and is a 'four point property'. So is Gromov's hyperbolicity. To formalize this, let us introduce for a metric space X the sets $K_m(X)$, which are the subsets of the $m \times m$ symmetric matrices comprising all the distances between pairs of points of X.

Open Problem: Give a sensible geometric condition on X that implies that $K_3(X)$ is as big as is allowed by the triangle inequality. See Gromov, Metric structures, p. 24

Gromov calls subsets of $K_m(X)$ 'curvature conditions'. Here is an example. Let $T \subseteq M_4(\mathbb{R})$ be the collection of all distances that can be realized by 4-tuples of points in a metric tree. One can characterize K_4 directly: it is the collection of those symmetric matrices $\{d_{ij}\}_{i,j=1}^4$, with zeros on the diagonal, such that of the three numbers

$$d_{12} + d_{34}, \quad d_{13} + d_{24}, \quad d_{14} + d_{23},$$

the greater two are equal. A geodesic space X with $K_4(X) \subseteq T$ is called a *real tree* or $\mathbb{R}$-*tree*. For example, consider the plane $\mathbb{R}^2$ with the following metric: $d((x_1, y_1), (x_2, y_2))$ equals $|y_1|+|y_2|+|x_1-x_2|$, except that if $x_1 = x_2$, then the distance equals $|y_1 - y_2|$. This is an $\mathbb{R}$-tree but not an ordinary tree.

(FIGURE)

(2.14) Lemma: *Let X and Y be compact metric spaces. If $K_m(X) = K_m(Y)$ for all m, then X and Y are isometric.*

Proof: (sketch) The condition implies that every finite subset of X isometrically embeds in Y. From this deduce that X itself isometrically embeds in Y (compare the proof of 2.4). Similarly Y isometrically embeds in X. Since any isometric embedding of a compact metric space into itself is surjective, these facts together imply that X and Y are isometric. ■

Since many geometric properties are defined in terms of the K_m, it appears that a notion of convergence respecting these will be interesting. We'll say that a sequence (or net) X_α of metric spaces converges to X *locally* if $K_m(X_\alpha)$ converges (Gromov-Hausdorff) to $K_m(X)$ for all m. This kind of convergence will preserve 'curvature conditions'.

2.2.3 Asymptotic Cones

Choose (!) an ultrafilter ω on $\mathbb{N}$. Let X_n be a sequence of metric spaces with assigned basepoints p_n. Let $\mathcal{X}$ denote the set of sequences $\{x_n\}$, $x_n \in X_n$, that have the property that $d_n(x_n, p_n)$ is a bounded function of n. On $\mathcal{X}$ we may define an equivalence relation $\sim$ by setting

$$\{x_n\} \sim \{y_n\} \quad \Leftrightarrow \quad \lim_{\omega} d_n(x_n, y_n) = 0.$$

Let $X_\omega = \mathcal{X}/\sim$. It is a metric space, with distance

$$d_\omega(\{x_n\}, \{y_n\}) = \lim_{\omega} d_n(x_n, y_n).$$

(2.15) Definition: *The metric space (X_ω, d_ω) defined above is called the* ultralimit *of the X_n associated to the ultrafilter ω, and is written* $\lim_\omega X_n$.

(2.16) Proposition: *Ultralimits are always complete.*

Proof: Let $\{x^i\}_{i=1}^\infty$ be a Cauchy sequence in X_ω, and suppose that each x^i is the ultralimit of a sequence $\{x^i_n\}_{n=1}^\infty$ in X_n. By induction one can construct a decreasing sequence of ω-thick subsets N_j of $\mathbb{N}$, with the property that for $1 \leqslant i, i' \leqslant j$ and $n \in N_j$ one has

$$|d_n(x^i_n, x^{i'}_n) - d_\omega(x^i, x^{i'})| < 2^{-j}.$$

Fix $y_n = x^j_n$, where j is the largest index for which $n \in N_j$. Then $\{y_n\}$ is a bounded sequence so it has an ultralimit y. We claim that $x^i \to y$ in the metric d_ω. Fix $\varepsilon > 0$. There is I such that $d_omega(x^i, x^{i'}) < \varepsilon$ for $i, i' \geqslant I$, and moreover $2^{-I} < \varepsilon$. It follows from the construction that $d_\omega(x^I, y) \leqslant 2\varepsilon$, so $d_\omega(x^i, y) \leqslant 3\varepsilon$ for all $i \geqslant I$. This suffices. ■

(2.17) Proposition: *Let $X_n = X$ be a constant sequence of proper metric spaces. Then the inclusion of constant sequences gives an identification of the ultralimit X_ω with X.*

Proof: To revert from the ultralimit to X, let $\{x_n\}$ be a sequence defining a point of X_ω. The sequence $\{x_n\}$ is bounded, so by properness it is precompact as a subset of X. Therefore, it has an ω-limit point in X, i.e. there is a point $x \in X$ such that, for every $\varepsilon > 0$, the set $\{n : |x_n - x| < \varepsilon\}$ is ω-thick. Now $\{x_n\}$ defines the same point of the ultralimit as the constant sequence $\{x\}$. ■

Properness is essential here — if each X is a countable discrete space, then X_ω is uncountable.

It is clear that the ultralimit respects 'first order properties' of metric spaces, such as the curvature conditions described above. Now, in the presence of completeness, being a geodesic space is equivalent to the *midpoint* *property*: for all x, y there exists z such that

$$d(x,z) = d(y,z) = \frac{1}{2}d(x,y).$$

Being a length space is equivalent to the *approximate midpoint property*: for all x, y there exists z such that

$$d(x,z) \leqslant \frac{1}{2}d(x,y) + \varepsilon, \quad d(y,z) \leqslant \frac{1}{2}d(x,y) + \varepsilon.$$

Both these properties are preserved by ultralimits. We conclude that the ultralimit of length spaces is a length space, and the ultralimit of geodesic spaces is a geodesic space.

(2.18) Remark: Suppose that the spaces X_n form a precompact family in the pointed Gromov-Hausdorff topology. Then the ultralimit X_ω is one of the (possibly many) Gromov-Hausdorff limit points of the sequence X_n. The proof is an exercise.

Let X_n and Y_n be two sequences of pointed metric spaces and let $f_n \colon X_n \to Y_n$ be a sequence of bornologous maps. Thus for every $R > 0$, $S_n(R)$ is finite, where

$$S_n(R) = \sup\{d(f_n(x_n), f_n(x_n')) : d(x_n, x_n') < R\}.$$

(2.19) Definition: *We will say that the sequence of maps* $\{f_n\}$ *is* equibornologous *if*

$$S(R) = \limsup_n S_n(R)$$

is finite for each $R > 0$. *We will say that it is* isobornologous *if, in addition,* $S(R) \to 0$ *as* $R \to 0$.

Example: Suppose that the f_n are (c_n, A_n) large scale Lipschitz. If the sequences $\{c_n\}$ and $\{A_n\}$ are bounded, then $\{f_n\}$ is equibornologous; if $\{c_n\}$ is bounded and $A_n \to 0$, then $\{f_n\}$ is isobornologous.

(2.20) Lemma: *Let* $f_n \colon X_n \to Y_n$ *be an isobornologous sequence of maps. Then the formula*

$$f_\omega(\{x_n\}) = \{f_n(x_n)\}$$

defines a continuous and coarse map $f_\omega \colon X_\omega \to Y_\omega$ *between the ultralimits* $X_\omega = \lim_\omega X_n$ *and* $Y_\omega = \lim_\omega Y_n$.

The proof is obvious.

Notice that if the maps $\{f_n\}$ only form an *equibornologous* sequence, then we may define a 'limit' map f_ω by setting $f_\omega(x) = \{f_n(x_n)\}$, where $\{x_n\}$ is an arbitrary choice[6] from the equivalence class representing x. The various possible choices of f_ω are all close to one another. In this case f_ω is coarse, but it isn't continuous. We could also define f_ω canonically as a map $X \times B \to Y$, where B is a very large bounded metric space, essentially the discrete space of all possible choices.

Now let X be a fixed metric space. Choose a basepoint (it doesn't matter which).

(2.21) DEFINITION: *The ultralimit of the sequence of pointed spaces* $\frac{1}{n}X$ *is called the* asymptotic cone *of* X, *and is denoted* $\mathrm{Cone}_\omega(X)$.

(Sometimes one replaces the sequence $\frac{1}{n}$ by another sequence tending to zero.)

The argument of the Lemma above shows that if $f\colon X \to Y$ is a (c, A)-large scale Lipschitz map, then it induces $\mathrm{Cone}_\omega(f)\colon \mathrm{Cone}_\omega(X) \to \mathrm{Cone}_\omega(Y)$, a c-Lipschitz (and therefore continuous) map between the asymptotic cones. Similarly a large scale Lipschitz equivalence induces a biLipschitz homeomorphism on the level of the asymptotic cones.

EXAMPLE: The asymptotic cone of $\mathbb{R}^n$ is $\mathbb{R}^n$. The same goes for any proper space that is equipped with a one-parameter group of self-similarities. The asymptotic cone of $\mathbb{Z}^n$ is therefore $\mathbb{R}^n$ as well. Notice that we hereby obtain a proof that $\mathbb{R}^n$ and $\mathbb{R}^m$ are coarsely equivalent if and only if $n = m$.

2.3 The Asymptotic Cone of Hyperbolic Space

In this section we are going to determine the asymptotic cone of the hyperbolic plane $\mathbb{H}^2$, and indeed of any space which shares the general qualitative features of $\mathbb{H}^2$ described by the 'thin triangles' property. In order to determine the relation of the thin triangles property to ultralimits, the first essential is to reformulate the property as a 'curvature property' in K_4, similar to our earlier reformulation as the property of being a geodesic space as the midpoint property, which belongs to K_3.

(definition of a tripod)

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[6] This involves a still more drastic application of the axiom of choice than we have so far made.

Let x, y, z be points in some metric space. There is a tripod (an abstract space) whose side lengths are exactly the distances $d(x,y)$, $d(y,z)$, and $d(z,x)$. We call it the *comparison tripod* for the triangle xyz.

(2.22) DEFINITION: *The* Gromov product $(y|z)_x$ *is the distance from* x *to the vertex of the comparison tripod; in other words,*

$$(y|z)_x = \tfrac{1}{2}(d(x,y) + d(x,z) - d(y,z)).$$

In any geodesic metric space, we will let $[y,z]$ denote the track of (any) geodesic segment from y to z.

(2.23) PROPOSITION: *Let* x, y, z *be points of a geodesic metric space. Then*

$$(y|z)_x \leqslant d(x,[y,z]),$$

the least distance from x *to the geodesic segment* $[y,z]$.

PROOF: Let $p \in [y,z]$ be a point where $d(x,[y,z])$ is realized, that is, $d(x,[y,z]) = d(x,p)$. Then $d(y,z) = d(y,p) + d(p,z)$. By the triangle inequality we have

$$d(x,y) \leqslant d(x,p) + d(y,p), \quad d(x,z) \leqslant d(x,p) + d(p,z).$$

Substituting these into the definition of the Gromov product we obtain the result. ■

We will see that in a metric space with the Rips property (a property that we have already observed to be true for the hyperbolic plane), one can also obtain an inequality running in the opposite direction. Recall

(2.24) DEFINITION: *A geodesic space* X *has the* Rips property *if there is a constant* $\delta > 0$ *such that, for any three points* $x, y, z \in X$ *and geodesic segments* $[x,y]$*,* $[y,z]$*, and* $[z,x]$*, all of* $[y,z]$ *is contained in a* δ*-neighborhood of* $[x,y] \cup [x,z]$.

(2.25) PROPOSITION: *A geodesic space* X *has the Rips property if and only if there is a constant* $\delta' > 0$ *such that each geodesic triangle in* X *(considered as a compact metric space in its own right) is* δ'*-close in the Gromov-Hausdorff metric*[7] *to the tripod with the same side lengths.*

---

[7] By this we understand that each side of the geodesic triangle is close to the corresponding side of the tripod?

PROOF: That the closeness condition implies the Rips property is apparent (?).

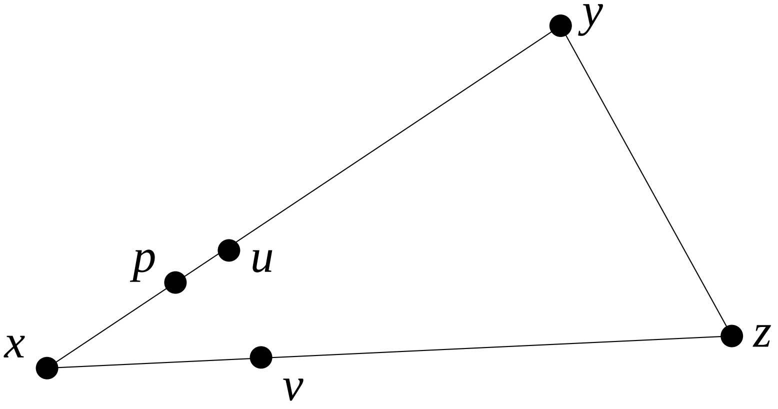
Consider the reverse implication, so suppose that X has the Rips property with constant δ. Let Δ be the a given geodesic triangle xyz in X, and let T_Δ be the unique tripod that has the same side-lengths as Δ. Let $f\colon \Delta \to T_\Delta$ be the natural map that is isometric on each side. Clearly, f is a 'short' map (that is, it decreases all distances).

We will show that for all $u, v \in \Delta$,

$$d(f(u), f(v)) \leqslant d(u, v) \leqslant d(f(u), f(v)) + 4\delta;$$

this will suffice to show that Δ and T_Δ are 8δ-Gromov-Hausdorff close (see Proposition ?).

Suppose the contrary. Then one easily sees that we may find $u, v \in \Delta$, say with $u \in [x, y]$ and $v \in [x, z]$, such that $f(u) = f(v)$ (which is to say, $d(x, u) = d(x, v) \leqslant (y|z)_x$) whereas $d(u, v) > 4\delta$. See the figure.



Let's first prove that $d(u, [x, z]) > 2\delta$. Indeed, the distance from u to $[x, z]$ is bounded below by the lesser of $(x|v)_u$ and $(v|z)_u$. But

$$(x|v)_u = \tfrac{1}{2}[d(x, u) + d(v, u) - d(x, v)] = \tfrac{1}{2}d(v, u) > 2\delta,$$

and

$$(v|z)_u = \tfrac{1}{2}[d(v, u) + d(z, u) - d(z, v)] \geqslant \tfrac{1}{2}d(v, u) > 2\delta,$$

since by the triangle inequality

$$d(z, u) \geqslant d(z, x) - d(x, u) = d(z, v).$$

Notice in particular that $d(u, x) > 2\delta$. Define p to be the point on $[x, u]$ such that $d(u, p) = \delta$. Then $d(p, [x, z] > \delta$. We shall show that $d(p, [y, z]) > \delta$ also. For

$$d(p, [y, z]) \geqslant d(x, [y, z]) - d(x, p) \geqslant (y|z)_x - d(x, p) \geqslant d(x, u) - d(x, p) = \delta.$$

We have shown therefore that $p \in [x, y]$ does not lie in a δ-neighborhood of $[x, z] \cup [z, y]$, contrary to the Rips property. The proof is thereby completed. ■

Henceforth we will style a geodesic space with the Rips property a *Gromov hyperbolic space.*

(2.26) COROLLARY: *In a Gromov hyperbolic space there is a constant* δ *such that, for all* x, y, z,

$$d(x, [y, z] - \delta \leqslant (y|z)_x \leqslant d(x, [y, z]).$$

Quadrilaterals

We are now going to consider 4-tuples of points.

(2.27) PROPOSITION: *Suppose* x, y, z, w *are four points in a metric tree. Then we have*

$$(x|z)_w \geqslant \min\{(x|y)_w, (y|z)_w\}.$$

PROOF: In a tree, all triangles are tripods, and thus $(x|y)_w = d(w, [x, y])$. The proposition therefore follows from the fact that the geodesic $[x, z]$ is contained in the union of the geodesics $[x, y]$ and $[y, z]$. ■

The inequality

$$(x|z)_w \geqslant \min\{(x|y)_w, (y|z)_w\} \tag{2.1}$$

of Proposition 2.27 defines a '4 point curvature condition' in the sense of Gromov (above). Not only is this condition satisfied for trees, but it is in fact *characteristic* of trees:

(2.28) PROPOSITION: *Let* X *be a (finite) metric space and suppose that every 4-tuple of points of* X *satisfies equation 2.1. Then* X *can be isometrically embedded in a tree.*

PROOF: Let $X = \{x_0, \dots, x_n\}$. Let $I_1, \dots, I_n$ be copies of the closed intervals $[0, d(x_0, x_1)], \dots, [0, d(x_0, x_n)]$ and let B be the disjoint union of the intervals I_k, so that B consists of pairs (t, k) where $t \in I_k$. Define a relation $\sim$ on B by

$$(t, k) \sim (t', k') \quad \Leftrightarrow \quad t = t' \leqslant (x_k | x_{k'})_{x_0}.$$

Thanks to the condition 2.1, this is an equivalence relation. Let $T = B/\sim$. Then T is a tree and one may introduce a metric into T by the formula

$$d((t, k), (t', k')) = t + t' - 2\min\{t, t', (x_k | x_{k'})_{x_0}\}.$$

The mapping which sends x_0 to $0 \in T$ and x_k to the endpoint of I_k is then the desired isometric embedding. ■

The restriction to *finite* X above is not really necessary. However, we need to loosen up our notion of 'tree' a bit.

(2.29) DEFINITION: *A* real tree *or* $\mathbb{R}$-tree *is a geodesic metric space which is uniquely arcwise connected: in other words, it is a geodesic space in which, for every* $x, y \in X$, *there is one and only one topologically embedded interval in* X *whose endpoints are* x *and* y. *(Necessarily, that interval is then a geodesic segment.)*

The proposition above then generalizes easily (exercise!) to

(2.30) PROPOSITION: *Any* $\mathbb{R}$*-tree satisfies equation 2.1 above, and a geodesic space satisfying 2.1 is an* $\mathbb{R}$*-tree. An arbitrary metric space* X *satisfies equation 2.1 if and only if it can be isometrically embedded in an* $\mathbb{R}$*-tree.*

(examples of $\mathbb{R}$-trees)

Let us return to the study of Gromov hyperbolic spaces.

(2.31) PROPOSITION: *In a Gromov hyperbolic space* X, *equation 2.1 is satisfied up to bounded error; that is, there is a constant* $\delta > 0$ *such that*

$$(x|z)_w \geqslant \min\{(x|y)_w, (y|z)_w\} - \delta$$

for all $x, y, z, w \in X$.

Proof: We can simply 'coarsify' the argument that we gave earlier to show that trees satisfy 2.1: in a hyperbolic space, the Gromov products such as $(x|z)_w$ are, up to bounded error, equal to the distances such as $d(w,[x,z])$ from w to the corresponding geodesic segments; and the segment $[x,z]$ is a bounded distance from $[x,y]\cup[y,z]$. The result follows. ■

It can be shown, conversely, that if a geodesic space satisfies the four-point condition above, then it satisfies the Rips condition (and so is Gromov-hyperbolic). We won't do that here, but see the book by Ghys and de la Harpe for an exhaustive account.

(2.32) Corollary: *Any asymptotic cone of a Gromov hyperbolic space is an $\mathbb{R}$-tree.*

Proof: If X satisfies the inequality of the proposition above, then $\frac{1}{n}X$ satisfies the inequality

$$(x|z)_w \geqslant \min\{(x|y)_w,(y|z)_w\} - \frac{\delta}{n}.$$

Thus the ultralimit $\mathrm{Cone}_\omega(X) = \lim_\omega \frac{1}{n}X$ is a geodesic space and satisfies 2.1. Thus it is an $\mathbb{R}$-tree. ■

In general the trees that arise in this way are very ramified. Let us say that an $\mathbb{R}$-tree X is *thoroughly branched* if there are at least three distinct geodesic rays (isometrically embedded copies of $\mathbb{R}^+$) originating at each point of X. Every asymptotic cone of $\mathbb{H}^2$ is thoroughly branched. Indeed, it is clear that there are uncountably many geodesic rays emanating from the origin; but $\mathrm{Cone}_\omega(\mathbb{H}^2)$ is a homogeneous space for the group G that is the ultraproduct[8] $\prod_\omega PGL(2,\mathbb{R})$ of countably many copies of the isometry group of $\mathbb{H}^2$; thus, there are uncountably many geodesic rays emanating from every point of $\mathrm{Cone}_\omega(\mathbb{H}^2)$.

2.4 Quasi-isometric rigidity

In this section we are going to work out an application of the machinery of asymptotic cones to the quasi-isometric rigidity of products. This application is due to Kapovich, Kleiner and Leeb (IHES Publications Mathématiques, volume 86, 1997, and Topology, volume 37, 1998). We will be looking only at a very special case of their constructions.

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[8] Exercise: Work out the definition.

One should take note that in their papers, KKL work with a more general notion of asymptotic cone where the base points and scale factors may vary. However, we can fix the base points if we restrict our attention to spaces admitting a cocompact group of isometries; and we can assume (I think) that the scale factors are a sequence of reciprocals of integers. So we can probably handle our special case with the notion of asymptotic cone we already defined.

Let X be a coarse space.

(2.33) Definition: *A* translation *of* X *is a bijective map* $h\colon X \to X$ *such that the graph of* h *is a controlled subset of* $X \times X$*. If* X *has a topology, we can speak of a* topological translation — *a translation that is also a homeomorphism.*

We will be concerned here with metric spaces, for which a translation is just a map that moves points by a uniformly bounded amount. We'll say that X is *topologically non-translatable* (TNT) if the group of topological translations is reduced to the identity.

(2.34) Proposition: *A thoroughly branched* $\mathbb{R}$*-tree is topologically nontranslatable.*

Proof: For any three points x_1, x_2, x_3 in an $\mathbb{R}$-tree X there is a unique topologically embedded tripod $T(x_1, x_2, x_3)$ with vertices at the three given points. Let $\sigma(x_1, x_2, x_3)$ denote the branch point of the tripod $T(x_1, x_2, x_3)$. Notice that, since T is topologically characterized, we have

$$h(\sigma(x_1, x_2, x_3)) = \sigma(h(x_1), h(x_2), h(x_3))$$

for any homeomorphism $h\colon X \to X$.

Observe (prove if you like) the following fact about metric trees: if a, b are two points in a tree with $d(a, b) = \ell$, if c is the midpoint of $[a, b]$, and if $d(a, a') \leqslant \ell/4$, $d(b, b') \leqslant \ell/4$, then the geodesic segment $[a', b']$ also passes through c.

Suppose now that $x \in X$, let h be a topological translation that moves points by no more than c. Pick three geodesic rays emanating from x and choose x_1, x_2, x_3 to be points on these three rays at distance $2c$ from x. Then $x = \sigma(x_1, x_2, x_3)$.

By the observation above, x lies on the geodesic segment $[h(x_1), h(x_2)]$, and similarly it lies on $[h(x_2), h(x_3)]$ and $[h(x_3), h(x_1)]$. Therefore

$$x = \sigma(h(x_1), h(x_2), h(x_3)) = h(x).$$

Since this is true for all x, h is the identity. ■

(2.35) DEFINITION: *A geodesic space* X *is* asymptotically nontranslatable *(ANT) if every asymptotic cone* $\mathrm{Cone}_\omega(X)$ *is TNT.*

As we have seen, for geodesic spaces, coarse equivalence is the same as large scale Lipschitz equivalence. We will need to control the constants appearing in such equivalences; for precision, let us say that two spaces X and Y are (c, A)-*equivalent* if there are (c, A)–large scale Lipschitz maps $X \to Y$ and $Y \to X$ whose composites are A-close to the identity.

(2.36) LEMMA: *Suppose that* X *is an asymptotically nontranslatable geodesic space. Then for every* (c, A) *there is a constant* $d = d(c, A)$ *such that any* (c, A)-*equivalence* $X \to X$ *has distance* $\leqslant d$ *from the identity.*

PROOF: Suppose not. Then for each i there is a $(c, A+1)$-equivalence $\varphi_i \colon X \to X$ such that $d(\varphi_i, 1) = n_i \to \infty$, $n_{i+1} > n_i$, and we may further assume that the distance from φ_i to the identity map is realized at some point of X. For simplicity, let's assume that this distance is always realized at the base-point x_0 of X, so that $d(\varphi_i(x_0), x_0) = n_i$. Now take the ultralimit

$$\mathrm{Cone}_\omega X = \lim_\omega \frac{1}{n_i} X.$$

Then $\varphi_\omega \colon \mathrm{Cone}_\omega(X) \to \mathrm{Cone}_\omega(X)$ is a topological translation that moves the basepoint, contrary to hypothesis.

How to deal with the 'for simplicity' hypothesis that we made above? If the distance from φ_i to the identity is realized at the points x_i, then we may form the asymptotic cone with the sequence $\{x_i\}$ of basepoints and proceed as before. Alternatively, if a cocompact group of isometries is acting, we may use this action to assume without loss of generality that x_i lies in some bounded neighborhood of x_0, and then the original construction works. ■

Here is the application to rigidity of products.

(2.37) THEOREM: *Let* X_1, X_2, Y_1, Y_2 *be ANT geodesic metric spaces. Suppose that* $\varphi \colon X_1 \times X_2 \to Y_1 \times Y_2$ *is a coarse equivalence which has the property that every asymptotic cone*

$$\mathrm{Cone}_\omega(\varphi) \colon \mathrm{Cone}_\omega(X_1) \times \mathrm{Cone}_\omega(X_2) \to \mathrm{Cone}_\omega(Y_1) \times \mathrm{Cone}_\omega(Y_2)$$

decomposes as a product of maps $\mathrm{Cone}_\omega(X_1) \to \mathrm{Cone}_\omega(Y_1)$ *and* $\mathrm{Cone}_\omega(X_2) \to \mathrm{Cone}_\omega(Y_2)$. *Then* φ *is at finite distance from a product of coarse equivalences.*

PROOF: Suppose that φ is a (c, A)-equivalence, and fix $\varepsilon < 1/c$. We will call a pair of points $x, x' \in X = X_1 \times X_2$ *horizontal* if their second coordinates agree, $\pi_2(x) = \pi_2(x')$, and *vertical* if their first coordinates agree. We will say that a horizontal pair x, x' has *nearly horizontal image* if

$$\frac{d(\pi_2(\varphi(x)), \pi_2(\varphi(x')))}{d(x, x')} \leqslant \varepsilon,$$

and we will say that it has *not very horizontal image* if

$$\varepsilon < \frac{d(\pi_2(\varphi(x)), \pi_2(\varphi(x')))}{d(x, x')} < \varepsilon^{-1}.$$

Similarly, we can define *nearly vertical image* and *not very vertical image*. We make the following

ASSERTION: *There exists a* $d_0 > 0$ *such that for all* $d \geqslant d_0$ *either*

(a) every d*-separated horizontal pair in* X *has nearly horizontal image in* Y*, or*

(b) every d*-separated horizontal pair in* X *has not very horizontal image in* Y

Moreover, there are similar statements where 'horizontal' is replaced by 'vertical'.

Grant this assertion for the moment. Let's note that it is not possible that the image of every d-separated horizontal pair should be both not very horizontal and not very vertical; this would contradict the claim that all ultralimits of φ preserve the product structure. By interchanging Y_1 and Y_2, if necessary, we may therefore suppose, without loss of generality, that (a) holds: every d-separated horizontal pair has nearly horizontal image. It follows that every d-separated horizontal pair has not very vertical image. (If every such pair had both nearly horizontal image and nearly vertical image, we would have a contradiction to the fact that φ is a (c, A)-equivalence. This statement amounts to saying that for each fixed $x_2 \in X_2$, the maps

$$\begin{tikzcd} X_1 \arrow[r, "i_{x_2}"] & X_1 \times X_2 \arrow[r, "\varphi"] & Y_1 \times Y_2 \arrow[r, "\pi_1"] & Y_1 \end{tikzcd}$$

are (c', A')-equivalences for some uniform constants (c', A'). Moreover, these maps are pairwise at bounded distance from one another (by the

Lipschitzness of φ). Since X_1 and Y_1 are ANT spaces, it follows from Lemma 2.36 that all of the maps are at bounded distance from some fixed coarse equivalence $\varphi_1 \colon X_1 \to Y_1$. Similarly all the partial maps

$$\begin{tikzcd} X_2 \arrow[r, "i_{x_1}"] & X_1 \times X_2 \arrow[r, "\varphi"] & Y_1 \times Y_2 \arrow[r, "\pi_2"] & Y_2 \end{tikzcd}$$

are at bounded distance from a coarse equivalence $\varphi_2 \colon X_2 \to Y_2$, and it follows easily that φ is at bounded distance from $\varphi_1 \times \varphi_2$.

It remains to prove the assertion. In fact, what we show is a local version: there is d_0 such that for all $d \geqslant d_0$, and every ball B in X of radius $10d$, either

(a) every d-separated horizontal pair in B has nearly horizontal image, or

(b) every d-separated horizontal pair in B has not very horizontal image.

From this, the global version of our assertion follows by a connectedness argument. But, if the local version is false then there are balls $B(p_k, 10d_k)$, $d_k \to \infty$, which contain d_k-separated horizontal pairs with nearly horizontal image and d_k-separated horizontal pairs with not very horizontal image. We deduce that for a suitable asymptotic cone the composite

$$\begin{tikzcd} \mathrm{Cone}_\omega(X) \arrow[r, "\varphi_\omega"] & \mathrm{Cone}_\omega(Y) \arrow[r, "\pi_2"] & \mathrm{Cone}_\omega(Y_2) \end{tikzcd}$$

is neither c-Lipschitz nor constant. This contradicts the assumption that the c-bi-Lipschitz map φ_ω preserves the product structure. ■

Our next objective is to prove topological rigidity for products of thoroughly branched trees:

(2.38) THEOREM: *Let* S_1, S_2, T_1, T_2 *be thoroughly branched trees. Then any homeomorphism* $S = S_1 \times S_2 \to T = T_1 \times T_2$ *is a product (possibly after interchanging factors).*

The proof requires a study of embedding topological disks (2-balls) in products of trees. Compare the proof of TNT for $\mathbb{R}$-trees which required the study of embedded 1-balls (intervals).

(2.39) DEFINITION: *A* quarter-flat *in a product* $T = T_1 \times T_2$ *of* $\mathbb{R}$*-trees is a product of two geodesic rays.*

(2.40) LEMMA: *Every topologically embedded disk in* T *is locally contained in the union of finitely many quarter-flats.*

('Locally' means that if, say, D is a closed topological disk in T and $D' \subset D$ is a subdisk whose closure is contained in the interior of D, then D' lies in the union of finitely many quarter-flats.)

The proof of the local finiteness lemma 2.40 relies on a homology calculation. We will give the proof later, but let us see how it implies Theorem 2.38. In order to prove the theorem, it will be sufficient to show that the horizntal and vertical geodesic rays in $T = T_1 \times T_2$ can be characterized purely in terms of the topology of that space[9]. That is what we shall do.

Let S, S' be subsets of T and let $z \in S$. We say that S, S' have the same *germ* at S if there is a neighborhood U of z in T for which $U \cap S = U \cap S'$. From lemma 2.40, if S is a topologically embedded 2-plane, and $z \in S$, then S has the germ at z of a finite union of quarter-flats. Indeed, there are finite subtrees F_1, F_2 of T_1, T_2 such that S has the germ at z of a subset of $F_1 \times F_2$. Since we are considering only germs, we may assume that each of F_1 and F_2 is a finite star (the cone on a finite set of points), branching only at $z_1 = \pi_1(z)$ and $z_2 = \pi_2(z)$ respectively. It is now clear that we may topologically characterize a germ of a quarter-flat at z: consider the class of all germs at z which are obtained as finite intersections of germs of topologically embedded 2-planes; a germ of a quarter-flat at z is a minimal 2-dimensional element of this class of germs. Furthermore, a germ of a horizontal or vertical geodesic is simply a 1-dimensional intersection of two germs of quarter-flats. This gives the desired topological characterization of horizontal and vertical geodesic rays, and so it completes the proof of Theorem 2.38.

From Theorem 2.38 and Theorem 2.37 there follows the main result:

(2.41) THEOREM: *Any coarse equivalence* $\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}^2 \times \mathbb{H}^2$ *is close to a product of coarse equivalences.*

It remains to prove Lemma 2.40. We use a homological argument. A basic construction is the operation of *straightening simplices* in a *CAT(0) space.*

We will define a geodesic space X to be a CAT(0) space if, for any geodesic triangle pqr in X, the midpoint m of qr is closer to p than it would be in an

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[9] Again, compare the 1-dimensional case: any topological embedding of $\mathbb{R}$ in a tree is a geodesic (up to reparameterization).

Euclidean triangle of this same side-lengths; that is to say

$$2d(m,p)^2 + \tfrac{1}{2}d(q,r)^2 \leqslant d(p,q)^2 + d(p,r)^2.$$

Exercise: any tree is a CAT(0) space.

The above formulation of the condition (due to Bruhat and Tits) has the advantage of rendering it obvious that the product of two CAT(0) spaces[10] is again a CAT(0) space. However, just as the midpoint condition for complete metric spaces implied the apparently stronger geodesic condition, so here a simple argument shows that all geodesic triangles in a CAT(0) space are thinner (corresponding points are closer together) than the corresponding triangles in the Euclidean plane. In particular *the metric on a CAT(0) space is convex:* if $\gamma_i(t)$, $i = 1,2$, are a pair of geodesics parametrized proportional to arc length, then

$$d(\gamma_1(t),\gamma_2(t)) \leqslant (1-t)d(\gamma_1(0),\gamma_2(0)) + td(\gamma_1(1),\gamma_2(1)).$$

This convexity of the metric has various topological implications. For instance, it implies that every CAT(0) space is contractible (radially to a basepoint). We will use a local version of this idea.

Let B be a subset of a Euclidean space E, and let $p \in E$ be a point affinely independent of B. The *cone* C_pB is the union of all line segments from p to points of B. Now suppose that $f\colon B \to X$ is a continuous map from B to a CAT(0) space X, and pick a point $q \in X$. We may extend f to a map

$$C_{p\mapsto q}(f)\colon C_pB \to X$$

by sending each line segment $[p,b]$ linearly onto the corresponding geodesic segment $[q,f(b)]$. Because the metric is convex, the map Cf is easily seen to be continuous.

Now let $\sigma\colon \Delta^q \to X$ be an ordered singular q-simplex in X. We define a new singular q-simplex $\mathrm{Str}(q)$, the *straightening* of q, by induction on the dimension: $\mathrm{Str}(\sigma) = \sigma$ for a 0-simplex, and if $q > 0$ then

$$\mathrm{Str}(\sigma) = C_{x_0\mapsto\sigma(x_0)}(\mathrm{Str}(\sigma_0))$$

where x_0 denotes the first vertex of the ordered simplex Δ^q, and $\sigma_0\colon \Delta^{q-1} \to X$ is the singular face opposite the vertex x_0.

(2.42) PROPOSITION: *Let* U *be any geodesically convex open subset of* X. *Then* Str *extends to a chain map from the singular chain complex of* U *to itself, and this chain map is chain homotopic to the identity.*

PROOF: Acyclic models. ■

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[10] We use the Pythagorean formula for the product metric.

The preceding discussion works in any CAT(0) space. Now, however, let us specialize to the CAT(0) space $T = T_1 \times T_2$, the product of two trees. Let us observe that, in T, the convex hull of two 2-flats is contained in the union of finitely many 2-flats; this is because, in a tree, the convex hull of two geodesics is contained in the union of finitely many geodesics. A simple induction thus tells us that for any singular cycle $\sigma \in C_\bullet(U)$, U an open subset of T, there is a finite union of 2-flats P (in particular, P is a 2-dimensional polyhedron) such that $[\sigma]$ lies in the image of the map

$$H_k(P \cap U) \to H_k(U)$$

defined on homology. From this construction it is easy to see

(2.43) LEMMA: *For all* $V \subseteq U \subseteq T$ *open, and all* $k > 2$, *the homology groups* $H_k(U, V)$ *vanish.*

Suppose now that D is a topologically embedded 2-disk in T. We will define the *homological support* of D to be the set of those points $z \in T \setminus \partial D$ such that the image of the fundamental class in $H_2(D, \partial D)$ under the map induced by inclusion,

$$H_2(D, \partial D) \to H_2(T, T \setminus \{z\}),$$

is nonzero. Clearly, from our earlier discussion, the homological support of D is locally contained in a finite union of 2-flats. Thus the proof is completed by

(2.44) LEMMA: *The homological support of* D *is precisely equal to the interior of* D.

PROOF: (exact sequences etc) ■

(compare the proof of the Jordan curve theorem).

3 Homological methods in coarse geometry

3.1 Definition of amenability

Let X be a metric space.

(3.1) DEFINITION: *A* uniform 0-chain *on* X *is a (possibly infinite) formal linear combination*

$$\sum_i \lambda_i [x_i]$$

where $\lambda_i \in \mathbb{R}$, $x_i \in X$, *and*

(a) The numbers λ_i *are uniformly bounded: the supremum* $\sup\{|\lambda_i|\}$ *is finite;*

(b) The subset $\{x_i\}$ *of* X *is* uniformly locally finite *(defined below).*

A subset $S \subseteq X$ is *uniformly locally finite* if for each $R > 0$ there is a number N_R such that $S \cap B(x; R)$ contains at most N_R points, for each ball $B(x; R)$ of radius R.

The collection of uniform 0-chains forms a vector space under the natural operations of addition and scalar multiplication: Formally speaking the easiest way to define these is to regard the chain $c = \sum_i \lambda_i [x_i]$ as defined by a function $\chi_c \colon X \to \mathbb{R}$, its *characteristic function*:

$$\chi_c(x) = \lambda_i \text{ if } x = x_i, \quad 0 \text{ otherwise.}$$

Then the addition and multiplication of chains is defined simply by the addition or multiplication of the corresponding functions. A function χ represents a chain if and only if it is bounded and its *support*

$$\operatorname{supp}(\chi) - \{x : \chi(x) \neq 0\}$$

is uniformly locally finite.

The vector space of uniform 0-chains on X is denoted $C_0^u(X)$.

EXAMPLE: Suppose that X is a finitely generated discrete group Γ equipped with a word length metric. Then X itself is uniformly locally finite. Consequently $C_0^u(X)$ is simply the space of all bounded functions on Γ — for historical reasons this space is usually denoted by $\ell^\infty(\Gamma)$.

EXAMPLE: Suppose that X is compact (or just bounded). Then uniformly locally finite subsets of X are actually finite. Thus $C_0^u(X)$ is just the apace of finite formal linear combinations of points of X; in other words it is the vector space with basis (corresponding to) the points of X.

(3.2) DEFINITION: *A uniform 0-chain* $c = \sum_i \lambda_i[x_i]$ *is* effective *if*

(a) Each coefficient λ_i *is* $\geqslant 1$*;*

(b) The set $\{x_i\}$ *is* coarsely dense *in* X.

By definition, $S \subseteq X$ *is coarsely dense if there exists a constant* A *such that every ball* $B(x; A)$ *of radius* A *in* X *meets at least one point of* S.

These notions behave well under coarse equivalence. Let $f\colon X \to Y$ and suppose that $c = \sum_i \lambda_i[x_i]$ is a uniform 0-chain in X. Then we can consider the formal linear combination

$$f_*(c) = \sum_i \lambda_i[f(x_i)]$$

of points of Y.

(3.3) PROPOSITION: *The formal linear combination* $f_*(c)$ *defines a uniform 0-chain in* Y*. If* c *is effective, then so is* $f_*(c)$.

PROOF: Let χ_c be the characteristic function of the chain c. The formal linear combination $f_*(c)$ corresponds to the characteristic function

$$\chi_{f_*(c)}\colon y \mapsto \sum_{\{x : f(x)=y\}} \chi_c(x).$$

Because f is a coarse equivalence, the sets $\{x : f(x) = y\}$ are uniformly bounded (independent of y), and so the uniform local finiteness of the chain c tells us that the sum appearing above contains a bounded number of terms (independent of y). Moreover each term in the sum is bounded in size, so $\chi_{f_*(c)}$ is a well-defined, bounded function. The support of $\chi_{f_*(c)}$ is contained in the image under f of the support of c; so it is uniformly locally finite[11]. Thus $f_*(c)$ is a uniform 0-chain on Y.

If c is effective, then every coefficient of $f_*(c)$ is a sum of coefficients of c, hence is at least 1. For the same reason, the support of $f_*(c)$ is exactly

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[11] It is easy to check that a coarse equivalence maps uniformly locally finite sets to uniformly locally finite sets.

equal to the image under f of the support of c; and it is easy to see that a coarse equivalence carries coarsely dense sets to coarsely dense sets. So, $f_*(c)$ is effective also. ■

We now define uniform 1-chains. (It would also be possible to define uniform chains of higher order, but we do not do so in this discussion.)

(3.4) DEFINITION: *A* uniform 1-chain *on* X *is a formal linear combination*

$$\sum_j \mu_j [y_j, z_j]$$

where $\mu_j \in \mathbb{R}$*, and* $y_j, z_j \in X$*, subject to the following conditions:*

(a) The numbers μ_j *are bounded;*

(b) The sets $\{y_j\}$ *and* $\{z_j\}$ *are uniformly locally finite;*

(c) The distances $d_X(y_j, z_j)$ *are bounded.*

Like a 0-chain, a 1-chain also has a characteristic function: in this case the function on $X \times X$ which is defined by

$$(y,z) \mapsto \mu_i \text{if } y = y_i \text{ and } z = z_i, \quad 0 \text{ otherwise.}$$

The characteristic function is bounded and has locally finite support which lies within an *entourage*, that is an R-neighborhood of the diagonal $\{(x,x) : x \in X\}$ in $X \times X$. Just as in the 0-chain case, this alternative representation makes it easy to make the 1-chains into a vector space, which we denote $C_1^u(X)$.

There is an important map relating 1-chains and 0-chains.

(3.5) DEFINITION: *The* boundary map *is the linear map* $b \colon C_1^u(X) \to C_0^u(X)$ *defined by*

$$b\left(\sum_j \mu_j [y_j, z_j]\right) = \sum_j \mu_j [y_j] - \sum_j \mu_j [z_j].$$

Informally speaking, b is defined on the "1-simplex" $[y,z]$ by $b([y,z]) = [y] - [z]$ and is then "extended by linearity".

(3.6) Definition:

- *(a) A* Ponzi scheme *on a metric space* X *is a uniform 1-chain whose boundary is an effective 0-chain.*
- *(b) A metric space* X *is called* amenable *if there are* no *Ponzi schemes on it. If it has any Ponzi schemes, it is called* nonamenable.

(3.7) Proposition: *Amenability is a coarse property: that is, if* X *and* Y *are coarsely equivalent, and* X *is amenable, then* Y *is amenable too.*

Proof: It is enough to show that if X has a Ponzi scheme, then Y has one too. We need to know that proposition 3.3 has a version for 1-chains: if $f\colon X \to Y$ is a coarse equivalence, and $c = \sum \mu_j[y_j, z_j]$ is a uniform 1-chain on X, then $f_*(c) = \sum \mu_j[f(y_j), f(z_j)]$ is a uniform 1-chain on Y. The proof is similar the one for 0-chains, and is left to the reader. But now suppose that c is a Ponzi scheme, so that $b(c)$ is effective. It is clear from the definitions that

$$b(f_*(c)) = \sum \mu_j([f(y_j)] - [f(z_j)]) = f_*(b(c))$$

and is therefore effective (by 3.3 again, since $b(c)$ is effective). Thus $f_*(c)$ is a Ponzi scheme on Y. ■

3.2 Examples of amenable and nonamenable spaces

Example: Every compact (or bounded) space must be amenable. Indeed, on any compact space X, 0-chains are finite, and therefore the linear map

$$\sigma\colon C_0^u(X) \to \mathbb{R}, \qquad \sigma\left(\sum \lambda_i[x_i]\right) = \sum \lambda_i$$

is well defined. Clearly $\sigma(b([y, z])) = \sigma([y] - [z]) = 0$ and hence by linearity $\sigma(b(c)) = 0$ for every 1-chain c. On the other hand, $\sigma(c') \geqslant 1$ for every effective 0-chain c'. It follows that $b(c)$ cannot be effective. Thus X has no Ponzi scheme, so it is amenable.

This elementary example can be generalized to some non compact spaces. Of course we cannot hope to sum a 0-chain over the whole of a non-compact space. Instead, we try to sum over larger and larger compact regions, and we hope that the errors introduced by not using the whole space will be asymptotically controllable.

(3.8) DEFINITION: *Let* X *be a metric space and let* V *be a compact (or bounded) subset of* X. *The linear map* $\sigma_V\colon C_0^u(X) \to \mathbb{R}$ *is defined by*

$$\sigma_V\left(\sum \lambda_i[x_i]\right) = \sum_{x_i \in V} \lambda_i.$$

The sum on the right in this definition is finite.

(3.9) DEFINITION: *Let* X *be a metric space,* $V \subseteq X$*, and let* $R > 0$*. The* R-boundary *of* V*, written* $\partial_R V$*, is the set*

$$\partial_R V = \{x \in X : B(x,R) \cap V \neq \emptyset,\ \ B(x,R) \cap (X \setminus V) \neq \emptyset\};$$

in other words, the set of those points of X *within distance* R *both of* V *and of its complement.*

We will need to be able to discuss the 'sizes' of regions like V and $\partial_R V$. There is a way of doing this in general metric spaces, based on the notions of *capacity* and *entropy* which we will discuss later; but for now, let us make the simplifying assumption that the space X itself is uniformly locally finite[12]. Then the 'size' of a set S can be defined simply as #S, the number of points in S.

Here are two key lemmas in extending the example above.

(3.10) LEMMA: *Let* c *be a uniform 1-chain on the uniformly locally finite space* X. *Then there is a constant* A *and* $R > 0$ *such that, for every* $V \subseteq X$,

$$|\sigma_V(b(c))| \leqslant A\#\partial_R V.$$

PROOF: Let c be given as $\sum_j \mu_j([y_j, z_j])$, and let R be the constant (part of the definition of a 1-chain) such that $d(y_j, z_j) < R$ for all j. Then

$$\sigma_V(b(c)) = \sum e_j \mu_j$$

where e_j is defined by the following table:

	$y_j \in V$	$y_j \notin V$
$z_j \in V$	0	−1
$z_j \notin V$	1	0

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[12] As we saw above, this is always true for the underlying metric space of a finitely generated discrete group.

In particular the contribution to the sum is zero unless one of y_j, z_j belongs to V and the other one does not; but in that case they both belong to $\partial_R V$. Thus the number of terms in the sum is at most $2\#\partial_R V$, and each one contributes at most $\sup|\mu_j|$ (in absolute value). By the triangle inequality the sum is bounded by $A\#\partial_R V$ where $A = 2\sup|\mu_j|$. ■

(3.11) LEMMA: *Let c' be an effective 0-chain on the uniformly locally finite space X. Then there is a constant B and $S > 0$ such that for every $V \subseteq X$*

$$\sigma_V(c') \geqslant B(\#V - \#\partial_S V)$$

PROOF: Choose $S = 2R$, where R is the constant provided by the definition of 'effective' such that every point of X is within distance R of the support of c'. Let $B' = \{x : B(x, S) \subseteq V\}$. Clearly, $\#B' \geqslant \#B - \#\partial_S B$.

By induction, choose a sequence x_k of points of B' such that each x_k does *not* belong to the union of the balls $B(x_l; S)$ for $l < k$. The induction finishes when the balls $B(x_l; S)$ (for $l = 1, \ldots, k$) cover B'. By uniform local finiteness, each ball of radius S has at most n members, for some constant n; so it takes at least $(1/n)\#B'$ of them to cover B'.

Now the distances $d(x_l, x_{l'})$ are at least S (by construction) when $l \neq l'$. Thus the balls $B(x_l, R)$ are all disjoint (remember that $R = S/2$). Each such ball contains a member of the support of c', and there are at least $(1/n)\#B'$ of them. Each member of the support of c' contributes at least 1 to the sum defining $\sigma_V(c')$. Thus

$$\sigma_V(c') \geqslant (1/n)\#B' \geqslant (1/n)(\#B - \#\partial_S B)$$

as was required. ■

Putting the two previous lemmas together we find that if the uniformly locally finite space X admits a Ponzi scheme — in other words, if it is nonamenable — then there must be a constant C and an $R > 0$ (we take R to be the greater of the numbers R and S appearing in the lemmas) such that

$$\frac{\#V}{\#\partial_R V} \leqslant K \tag{3.2}$$

for *every* compact $V \subseteq R$. (The constant K equals $B/A + 1$.)

(3.12) THEOREM: *If* X *(still assumed to be uniformly locally finite) is nonamenable, then for each* $x \in X$ *there is a constant* $\alpha > 1$ *such that* $\#B(x, r) \geqslant \alpha^r$ *when* r *is sufficiently large.*

Briefly, we say that a nonamenable space has *exponential volume growth*. The converse of this theorem is not true in general: there exist examples of amenable spaces (even groups) having exponential growth.

PROOF: Let the constants K and R be as in equation 3.2 above. For simplicity of notation let $B_r = B(x, r)$. Notice that $\partial_R B_{r+R}$ is contained in B_{r+2R} and does not meet B_r; so

$$\#B_{r+2R} \geqslant \#B_r + \#\partial_R B_{r+R} \geqslant \#B_r + (1/K)\#B_{r+R} \geqslant (1 + 1/K)\#B_r.$$

By induction

$$\#B_{2nR} \geqslant (1 + 1/K)^n$$

which gives exponential growth. ■

EXAMPLE: The number of points in a 'ball' of radius r in the group $\mathbb{Z}$ of integers (with its natural metric) is at most $2r+1$. This function is certainly not of exponential growth. Thus, $\mathbb{Z}$ is an amenable group.

EXAMPLE: Generalizing the above, the group $\mathbb{Z}^n$ has growth which is polynomial of order n. Indeed, the number of points in a ball of radius r in $\mathbb{Z}^n$ is bounded by the volume of the ball of radius $r + 1$ in Euclidean n-space; since each lattice point can be thought of as defining a unit cube, centered at that point; and the cubes corresponding to distinct lattice points are disjoint. Hence, $\mathbb{Z}^n$ is amenable.

EXAMPLE: It follows that $\mathbb{R}^n$ is amenable too, since amenability is a coarse invariant. Moreover, by the Milnor-Wolf theorem, any finitely generated group which acts properly and cocompactly on $\mathbb{R}^n$ is amenable too.

We now give some nonamenable examples.

EXAMPLE: The free group on two[13] generators $\{g, h\}$ is a nonamenable group. To show this we must construct a Ponzi scheme. Our 1-chain c is the sum of all those 1-simplices $[y, z]$ such that $d(y, z) = 1$ (which means that $z^{-1}y \in \{g^{\pm 1}, h^{\pm 1}\}$) and $d(y, e) < d(z, e)$ (that is, y is strictly closer to the identity, in word length, than z is). What is the boundary of this

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[13] Or more! but we stick to 2 for simplicity.

1-chain? Any group element x has four neighbors at distance 1 from it, namely the elements xg, xg^{-1}, xh, and xh^{-1}; and of these four elements, in general[14] three will be further away from the identity and one will be closer: the closer one is xl^{-1} where l is the last letter in the word representing x. Thus the coefficient of $[x]$ in $b(c)$ will be a sum of four terms ± 1, at least three of which are positive; so it will be at least 2. Thus $b(c)$ is effective.

EXAMPLE: The hyperbolic plane H^2 is nonamenable. Here is a geometric proof. Choose an origin O in H^2. Let Γ be a discrete group of isometries acting freely on H^2 with quotient a compact surface, and let P be a Dirichlet polygon for Γ. Then the copies gP ($g \in \Gamma$) of P cover H^2 and meet only along the edges — in other words, they *tile* the plane. Pick a point $p \in P$; then $X = \{gp : g \in \Gamma\}$ is uniformly locally finite (by a volume argument like that we used for euclidean space above) and coarsely dense (as P is compact). Our Ponzi scheme is given by a 1-chain c defined as follows: a 1-simplex $[y, z]$ appears in the Ponzi scheme if $y = gp$ and $z = hp$ both belong to X, and the corresponding copies gP, hP of the polygon P are adjacent along an edge $e = [a, b]$, with y lying on the same side of e as the origin. The coefficient of $[y, z]$ is the area of the triangle $Oe = Oab$ subtended by the edge e at the origin. Since all triangles in hyperbolic space have area at most π, this defines a uniform 1-chain. We now investigate $b(c)$. The coefficient with which $[x]$ appears in $b(c)$ (where $x = gp \in X$) is equal to the sum, with appropriate signs, of the areas of the triangles Oe, as e runs over the edges of gP. But this sum is simply the area of gP itself, which is equal to the area of P as g is an isometry. The area of P is equal to $-2\pi\chi(H^2/\Gamma)$ (where χ denotes the Euler characteristic) which is certainly greater than 1; so $b(c)$ is effective as required.

Using the Milnor-Wolf theorem we find that every group of isometries acting properly and cocompactly on the hyperbolic plane is nonamenable. This includes, for example, all the groups Γ such that H^2/Γ is a compact hyperbolic surface.

3.3 Amenable Groups

Now we will relate amenability a bit more closely to group theory.

Let Γ be a finitely generated group (which, as usual, we consider as a metric space). We'll introduce a slightly different notation for 1-simplices in

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[14] If x *is* the identity all four will be further away.

Γ; we will write the 1-simplex $[y, z]$ as $[y] \otimes [g]$, where g is the group element $z^{-1}y$. Obviously y and z can be reconstructed from y and g, so that the new notation is equivalent to the old. The advantage of the new notation is this: when we write a uniform 1-chain as

$$c = \sum_j \mu_j [y_j, z_j] = \sum_j [\mu_j] \otimes g_j$$

the requirement that the distances $d(y_j, z_j)$ be bounded translates into the requirement that the g_j lie within a bounded distance of the identity element of Γ; in other words, the set $\{g_j\}$ is *finite.* Thus c can be written as a finite sum of 1-chains of the form

$$c_g = \sum_{\{g_j = g\}} \mu_j [y_j] \otimes g$$

with fixed g.

The boundary of c_g is a 0-chain of the form $\sum \mu_j([y_j] - [y_j g^{-1}])$, and its characteristic function is $x \mapsto f(x) - f(xg)$. Thus we have proved

(3.13) Proposition: *The range of the boundary map* $b \colon C_1^u(\Gamma) \to C_u^0(\Gamma)$ *consists of the subspace of* $\ell^\infty(\Gamma)$ *spanned by the functions of the form*

$$x \mapsto f(x) - f(xg)$$

for $g \in \Gamma$, $f \in \ell^\infty(\Gamma)$.

Let V be a vector space of bounded real-valued functions (on some set). A *mean* on V is a linear map $m \colon V \to \mathbb{R}$ such that

$$\inf f \leqslant m(f) \leqslant \sup f$$

for all $f \in V$. (It is really only necessary to require the inequality for the supremum of f; the one for the infimum follows by applying the other one to $-f$ and using linearity.) If $V = \ell^\infty(\Gamma)$ we say that a mean m is *invariant* if $m(f) = m(f \circ R_g)$, for all $f \in V$ and all $g \in \Gamma$. Here $R_g \colon G \to G$ is the map given by right multiplication by g, that is, $R_g(h) = hg$. An invariant mean assigns the same 'mean value' to f and to f 'translated' by the group action.

Example: We probably believe that, on average, half of all integers are even. Here is a justification. Let m be an invariant mean on the group $\mathbb{Z}$ and let f be the function such that $f(n)$ equals 1 if n is even and 0 if n is odd. Then the average proportion of even integers can be described by the mean $m(f)$. The function $1 - f$ is equal to $f \circ R_g$, where $g = +1$. Hence $m(f) = m(1 - f)$. But by linearity

$$m(f) + m(1 - f) = m(1) = 1$$

(the last equality follows from the definition of a mean since the constant function 1 has both supremum and infimum equal to 1). Hence $m(f) = \frac{1}{2}$, as we expected.

This argument assumed that $\mathbb{Z}$ had at least one invariant mean. How can the existence of invariant means be checked? We will prove

(3.14) THEOREM: *A group Γ has an invariant mean if and only if it is amenable.*

This is the reason for the terminology 'amenable'. One direction of the proof is easy. It follows from the next two lemmas.

(3.15) LEMMA: *If m is an invariant mean on a group Γ, then $m(bc) = 0$ for every uniform 1-chain c.*

PROOF: By lemma 3.13 above, bc is a finite linear combination of terms of the form $f - f \circ R_g$, each of which has mean value 0 by invariance. ■

(3.16) LEMMA: *For any invariant mean m on Γ, $m(c) > 0$ if c is an effective 0-chain.*

PROOF: Let $c = \sum \lambda_i [x_i]$ be an effective 0-chain. Then there is $R > 0$ such that every element of Γ is within distance R of one of the $[x_i]$. Let $F = B(e, R)$, the ball of radius R around the identity element; then every element of Γ can be written $x_i g$, for some x_i in the support of c and some $g \in F$. Hence

$$c^* = \sum_i \sum_{g \in F} \lambda_i [x_i g]$$

is a 0-chain which has value $\geqslant 1$ at each point of Γ; so $m(c^*) \geqslant 1$. But $m(c^*) = m(c) \# F$ by invariance, hence $m(c) > 0$. ■

To prove Theorem 3.14 in the other direction, take V as $\ell^\infty(\Gamma)$, and W as the subspace consisting of the boundaries of uniform 1-chains. Suppose that $f \in W$. I claim that $\inf f \leqslant 0$. If not, then $f \geqslant \delta > 0$ for some δ; and then $(1/\delta) f \geqslant 1$ is an effective 0-chain and is the boundary of a 1-chain, contrary to the assumption of amenability. Applying the same argument to $-f$ instead we find that $\sup f \geqslant 0$. Thus the zero functional, $f \mapsto 0$ for all $f \in W$, is a mean on W. By the Hahn-Banach Theorem, there is a mean m on $\ell^\infty(\Gamma)$ which restricts to zero on W. But such a mean is invariant, since W contains all the differences $f - f \circ R_g$.

4 Asymptotic dimension

4.1 Definitions

We are going to investigate certain properties of coarse spaces analogous to the topological covering dimension of compact metric spaces. (See the book by Hurewicz and Wallman.) The basic definitions can be given for any coarse space, but we will soon specialize to the case of a metric space — usually a proper geodesic space.

(4.1) Definition: *Let* X *be a coarse space, and let* $U \subset X \times X$ *be a controlled set. We will say that* $D \subseteq X$ *is* U-disconnected *if one can write it as a disjoint union* $D = \bigsqcup_{i=1}^{\infty} D_i$ *such that*

(a) $\bigcup D_i \times D_i = W$ *is controlled, and*

(b) For $i \neq j$, $D_i \times D_j$ *is disjoint from* U.

We refer to D *as* W-bounded *(property (a)) and* U-disjoint *(property (b)).*

In the case of metric spaces we'll modify the terminology in the obvious way, speaking of an d-*disjoint* and R-*bounded* set D.

(4.2) Definition: *The space* X *has* asymptotic dimension zero *if it is* U-*disconnected for every controlled set* U.

Example: Every bounded metric space has asymptotic dimension zero. No non-compact geodesic metric space has asymptotic dimension zero. The space $X = \{n^2 : n \in \mathbb{N}\}$, with the metric it inherits as a subspace of $\mathbb{N}$, has asymptotic dimension zero.

(4.3) Definition: *The space* X *has* asymptotic dimension $\leqslant n$ *if, for every controlled* U, *it can be written as the union of at most* $n+1$ U-*disconnected subsets.*

Asymptotic dimension is invariant under coarse equivalence. Indeed, suppose that X has asymptotic dimension $\leqslant n$ and that $f\colon X \to X'$ is a coarse equivalence. Let U' be a controlled set for X'. Because f is a coarse equivalence, there is a controlled set U for X such that $f_*(U) \supseteq U'$. There is a controlled set W for X such that X can be written as the union of $n+1$ subsets $D_0, \ldots, D_n$ which are U-disjoint and W-bounded; and now their images $f(D_0), \ldots, f(D_n)$ are U'-disjoint and W'-bounded, where W' is the controlled set $f_*(W)$.

(4.4) Remark: It is not true that asymptotic dimension is monotonic decreasing under surjective coarse maps, any more than the corresponding thing is true for topological dimension; think of the Peano curve.

Example: $\mathbb{R}$ has asymptotic dimension $\leqslant 1$. It is easy to draw a picture to convince yourself that $\mathbb{R}^2$ has asymptotic dimension $\leqslant 2$, and in fact we shall soon see that $\mathbb{R}^n$ has asymptotic dimension exactly n. (By this we mean that it has asymptotic dimension $\leqslant n$ and doesn't have asymptotic dimension $\leqslant n-1$.)

Example: Let X be a proper metric space, and equip X with the *topological* coarse structure coming from its one-point compactification. Then X has asymptotic dimension one. To see this observe that a subset $U \subseteq X \times X$ is controlled if and only if it is *proper*, meaning that $U[K] := \{y : \exists x \in K : (x, y) \in U\}$ and $U^*[K] := \{y : \exists x \in K : (y, x) \in U\}$ are precompact whenever K is precompact.

Now let U be a controlled set for X. Choose a point $x \in X$ and define precompact subsets $K_0, K_1, \ldots$ of X inductively by

$$K_0 = \{x\}, \quad K_{n+1} = U[K_n].$$

Without loss of generality we may assume that $\bigcup_n K_n = \bigcup_n K_n^\circ = X$ (if not, enlarge U as necessary). Put

$$D_0 = \bigsqcup_{n \text{ even}} K_n \setminus K_{n-1}, \quad D_1 = \bigsqcup_{n \text{ odd}} K_n \setminus K_{n-1}.$$

These are each U-disconnected sets (notice that $\bigcup (K_n \setminus K_{n-1}) \times (K_n \setminus K_{n-1})$ is controlled), and this completes the proof.

(4.5) Proposition: *Any metric tree (even an $\mathbb{R}$-tree) has asymptotic dimension 1.*

Proof: Fix an origin e in the tree T, and recall the characteristic property of a tree that

$$(x|z) \geqslant \min\{(x|y), (y|z)\},$$

where the Gromov products are taken with respect to e, so that

$$(x|y) := \tfrac{1}{2}(d(x, e) + d(y, e) - d(x, y)).$$

Let d be given and define the *annuli*

$$A_k = \{x \in T : kd \leqslant d(x,e) < (k+1)d\}.$$

We shall show that $\bigsqcup_{k \text{ even}} A_k$ and $\bigsqcup_{k \text{ odd}} A_k$ are d-disconnected. For this purpose we have to decompose these unions into d-disjoint pieces of uniformly bounded size. In fact, we shall decompose each annulus A_k into d-disjoint pieces of diameter at most $3d$. Since A_k and A_l are themselves at distance d from each other whenever $|k-l| \geqslant 2$, this will suffice.

Define a relation $\sim$ on A_k by

$$x \sim y \quad \Leftrightarrow \quad (x|y) \geqslant (k - \tfrac{1}{2})d.$$

Thanks to the characteristic property of a tree, recalled above, this is an equivalence relation. Each equivalence class has diameter $\leqslant 3d$ because if $x \sim y$ then

$$d(x,y) = d(x,e) + d(y,e) - 2(x|y) \leqslant 2(k+1)d - 2(k - \tfrac{1}{2})d = 3d.$$

Different equivalence classes are d-disjoint because if $x \not\sim y$ then

$$d(x,y) = d(x,e) + d(y,e) - 2(x|y) \geqslant 2kd - 2(k - \tfrac{1}{2})d = d.$$

The proof is completed. ■

In the next proposition we are going to give some reformulations of the notion of finite asymptotic dimension. One of these reformulations will depend on the metrization of an abstract simplicial complex. Let S be a simplicial complex with vertex set V. Let H denote the Hilbert space $\ell^2(V)$. The map which sends each vertex v to the corresponding unit vector $e_v \in H$ extends to an embedding of the geometric realization $|S|$ of S into the unit ball of H. By means of this map, $|S|$ inherits a metric from H; this metric is called the *metric of barycentric coordinates.* In this metric each simplex inherits its standard euclidean metric, but $|S|$ overall is bounded.

There is also another metric we can give to $|S|$, the *intrinsic metric.* This may be defined as the unique length metric which agrees on each simplex with the natural Euclidean metric of that simplex. It follows from Bridson's thesis that, if S is finite-dimensional, then $|S|$ is a complete geodesic space in the intrinsic metric. (This is easier to prove when S is locally finite, and that case will suffice for our purposes here.)

This whole business of metrizing simplicial complexes is more complicated than it may at first appear. See Dowker, *American J. Math* **74**(1952), 555–577.

Let $f\colon X \to |S|$ be a map from a metric space X to the geometric realization of an abstract simplicial complex S. Following Gromov, we say that f is *uniformly cobounded* if there is a uniform bound for the diameter of the inverse image, under f, of the star of any vertex of $|S|$. If S is locally finite and $|S|$ is equipped with the intrinsic metric, this is the same as to say that f is effectively proper (compare 1.3).

(4.6) THEOREM: *Let X be a proper metric space. Then the following are equivalent:*

- *(a) X has asymptotic dimension $\leqslant n$;*
- *(b) X admits an anti-Čech system comprised of coverings all of whose nerves have dimension $\leqslant n$;*
- *(c) For each $\varepsilon > 0$ there is an ε-Lipschitz and uniformly cobounded map from X to an n-dimensional polyhedron (the geometric realization of an n-dimensional simplicial complex), equipped with the metric of barycentric coordinates.*

If X is a geodesic space these conditions are also equivalent to

- *(d) For each $\varepsilon > 0$ there is an ε-Lipschitz and effectively proper map from X to an n-dimensional polyhedron equipped with the intrinsic metric.*

PROOF: **(a) implies (b):** Let $\ell > 0$ be given. By definition of asymptotic dimension, X can be covered by $n+1$ 6ℓ-disconnected sets $D_0, \ldots, D_n$, each of which is a union of d-disjoint and uniformly bounded pieces, $D_i = \bigsqcup_j D_i^j$, $i = 0, \ldots, n$. Let Y_i^j be the 2ℓ-neighborhood of D_i^j. The sets Y_i^j form a cover $\mathcal{U}$ of X which is uniformly bounded and has Lebesgue number at least ℓ. Moreover, $Y_i^j \cap Y_i^{j'} = \emptyset$ if $j \neq j'$, and it follows that the geometric realization $|\mathcal{U}|$ has dimension at most n. Thus X has uniformly bounded n-dimensional covers with arbitrarily large Lebesgue number, which proves (b).

(b) implies (c): Begin by recalling the following standard construction. Suppose that $\mathcal{U}$ is an open cover of a space X, and let $\{\varphi_U : U \in \mathcal{U}\}$ be a locally finite partition of unity subordinate to $\mathcal{U}$. Let $N(\mathcal{U})$ denote the nerve of $\mathcal{U}$, and let $|N(\mathcal{U})|$ be its geometric realization, which we may regard as a subset of the Hilbert space $H = \ell^2(\mathcal{U})$. The map

$$\Phi\colon x \mapsto \sum_U \varphi_U(x)e_U \in H$$

is continuous from X to the polyhedron $|N(\mathcal{U})|$. Suppose moreover that $|\mathcal{U}$ is n-dimensional and that each φ_U is ε-Lipschitz. Then Φ is $(2n+2)\varepsilon$-Lipschitz, considered as a map to $|N(\mathcal{U})|$ with the metric of barycentric coordinates; this is because for any $x, x' \in X$ there can be at most $2n+2$ sets U for which $\varphi_U(x) \neq \varphi_U(x')$.

Having made these observations, suppose (b). Let $\varepsilon > 0$ be given, and let $\ell = (4n+5)\varepsilon^{-1}$. Let $\mathcal{U}$ be a uniformly bounded, locally finite, n-dimensional open cover of X with Lebesgue number greater than ℓ. It is a straightforward matter to construct a partition of unity subordinate to $\mathcal{U}$ all of whose constitutent functions are $\varepsilon/(2n+2)$-Lipschitz. (Construction: Let ψ_U be the function which is 0 outside U and, for $x \in U$, $\psi_U(x)$ is $\min\{1, \varepsilon/(2n+2)d(x, X \setminus U)\}$. The Lebesgue number hypothesis ensures that $\Sigma(x) = \sum \psi_U(x) \geqslant 1$ for all x. Put $\varphi_U(x) = \psi_U(x)/\Sigma(x)$.) It follows that $\Phi\colon X \to |N(\mathcal{U})|$ is ε-Lipschitz. Moreover, the inverse image under Φ of the star of any vertex is contained in the member of $\mathcal{U}$ corresponding to that vertex; hence it is uniformly bounded. Thus Φ satisfies the requirements of (c).

(c) implies (a): Let P be an n-dimensional polyhedron, equipped with the metric of barycentric coordinates. Define subsets $D_0, \ldots, D_n$ of P as follows: D_i is the union of the stars, relative to the second barycentric subdivision of P, of those vertices of the first barycentric subdivision that correspond to i-dimensional simplices of P. Elementary geometry shows that there is a constant $\delta > 0$ (depending only on n) such that each D_i is the disjoint union of pieces D_i^j (the individual stars), each of which is contained in the star of some vertex of the original polyhedron P, and such that distinct pieces are separated (in the metric of barycentric coordinates) by at least δ.

Suppose now that $\Phi\colon X \to P$ is ε-Lipschitz and uniformly cobounded. Then $\Phi^{-1}(D_i)$ is a δ/ε-disconnected subset of X, and thus X can be written as the union of $n+1$ δ/ε-disconnected subsets. It is now clear that (c) implies (a).

We leave to the reader the check that (c) and (d) are equivalent in the gedoesic case. ■

4.2 Composition properties of asymptotic dimension

Asymptotic dimension has various 'composition' properties:

(4.7) PROPOSITION: *If* $Y \subseteq X$*, then* $\operatorname{asdim} Y \leqslant \operatorname{asdim} X$.

The proof is obvious, since if D is d-disconnected in X, then $D \cap Y$ is d-disconnected in Y.

(4.8) PROPOSITION: *Let* X *and* Y *be metric spaces. Then*

$$\operatorname{asdim}(X \times Y) \leqslant \operatorname{asdim}(X) + \operatorname{asdim}(Y).$$

This follows easily from the characterization (c) of asymptotic dimension in theorem 4.6.(details?) Notice that we probably shouldn't expect equality here; compare the example of Pontrjagin of two compact 2-dimensional spaces whose product is 3-dimensional. Can one give an explicit example?

It follows of course that $\operatorname{asdim} \mathbb{R}^n \leqslant n$. Let us show that we do indeed have equality here; the simplest way to do that is to consider the relation of asymptotic dimension to coarse cohomology.

Begin by recalling the Milnor exact sequence for coarse cohomology:

$$\begin{tikzcd} 0 \arrow[r] & \lim^1 H_c^{q-1}(|\mathcal{U}_j|; R) \arrow[r] & HX^q(X; R) \arrow[r] & \lim H_c^q(|\mathcal{U}_j|; R) \arrow[r] & 0 \end{tikzcd}$$

All of the terms in this exact sequence are coarse invaraints. Let us call the space X *tame* if the $\lim^1$ term is zero.

(4.9) LEMMA: *Let* X *be a metric space. If* $q > \operatorname{asdim} X + 1$ *then* $HX^q(X; R) = 0$*. If* X *is tame, the same conclusion follows when* $q > \operatorname{asdim} X$.

PROOF: Using (b) of theorem 4.6, choose an anti-Čech system $\mathcal{U}_j$ consisting of n-dimensional covers, and observe that, for these covers, $H_c^q(|\mathcal{U}_j|; R) = 0$ when $q > n$. ■

The example of $X = \{n^2 : n \in \mathbb{N}\}$ shows that we cannot omit the tameness condition. Indeed, we have already seen that this space is 0-dimensional; but $HX^1(X); R)$ is very large (in fact, any map $f \colon X \to R$ defines a coarse 1-cocycle $\varphi(x^0, x^1) = f(x^1) - f(x^0)$, and this 1-cocycle is a 1-coboundary only if f is constant outside a compact set). Here all the contribution to HX^1 comes from the $\lim^1$ term in the Milnor exact sequence.

Since $\mathbb{R}^n$ is tame and $HX^n(\mathbb{R}^n)$ is nonzero, we find that $\operatorname{asdim} \mathbb{R}^n \geqslant n$, and combining this with 4.8 we have equality.

(4.10) PROPOSITION: *Let* $X = X' \cup X''$ *be a metric space. Then*

$$\operatorname{asdim}(X) = \max\{\operatorname{asdim}(X'), \operatorname{asdim}(X'')\}.$$

Contrast the case of classical (topological) dimension where $\dim(X' \cup X'') \leqslant \dim X' + \dim X'' + 1$.)

PROOF: (Bell–Dranishnikov) We make use of the following notion. Let $\mathcal{U}$ and $\mathcal{V}$ be two families of subsets of a metric space. For $V \in \mathcal{V}$ and $r > 0$ define

$$N_r(V;\mathcal{U}) = V \cup \bigcup\{U \in \mathcal{U} : d(U,V) \leqslant r\},$$

that is the union of V with those members of $\mathcal{U}$ that are within distance r of it. Fixing $\mathcal{U}$ and $\mathcal{V}$ define their r-*expanded union* $\mathcal{W}$ to be the collection of all the $N_r(V;\mathcal{U})$ together with all those $U \in \mathcal{U}$ that are at distance greater than r from every $V \in \mathcal{V}$. Notice that, by construction, $\bigcup \mathcal{W} = \bigcup \mathcal{U} \cup \bigcup \mathcal{V}$.

It is simple to verify the following CLAIM: *Suppose that* $\mathcal{U}$ *is* r-*disjoint and* R-*bounded (with* $R \geqslant d$*), and that* $\mathcal{V}$ *is* 5R-*disjoint and* S-*bounded. Then their* r-*expanded union* $\mathcal{W}$ *is* r-*disjoint and* $(S + 2r + 2R)$-*bounded.*

Now suppose that X' and X'' have asymptotic dimension $\leqslant n$. Then for each $r > 0$ there is an R such that X' can be covered by $(n + 1)$ r-disjoint, R-bounded families $\mathcal{U}_0, \ldots, \mathcal{U}_n$, and there is an S such that X'' can be covered by $(n+1)$ 5R-disjoint and and S-bounded families $\mathcal{V}_0, \ldots, \mathcal{V}_n$. By the claim, the r-expanded unions $\mathcal{W}_i$ of $\mathcal{V}_i$ and $\mathcal{U}_i$ are r-disjoint bounded families and cover X. ■

As pointed out by Bell and Dranishnikov, one can generalize this construction to certain infinite unions. Suppose that $\{X_\alpha\}$ is a family of metric spaces. There is an evident definition of what it means that the fanily should have *asymptotic dimension* $\leqslant n$ *uniformly*; namely that for each $r > 0$ there should exist $R > 0$ such that each X_α can be covered by at most $(n+1)$ r-disjoint, R-bounded families.

EXAMPLE: Associated to a statement such as $\operatorname{asdim} X \leqslant k$ there is a finer invariant, the so-called *type function*. This is the function $\tau_{k,X} \colon \mathbb{R}^+ \to \mathbb{R}^+$ defined as follows: $\tau(r)$ is the infimum of those $R > 0$ for which X can be covered by $(k + 1)$ r-disjoint, R-bounded families. If $\tau(r) \leqslant a(r + 1)$ for some constant $a > 0$, we say that X has finite asymptotic dimension *of linear type*. Euclidean spaces and free groups have finite asymptotic dimension of linear type; the reader may check that the constructions so far discussed in this section preserve linear type.

Suppose now that X has finite asymptotic dimension $\leqslant k$, of linear type, and let $X_n = \frac{1}{n}X$ be X with rescaled metric (cf. earlier). The family of spaces $\{X_n\}$ then has asymptotic dimension $\leqslant k$ uniformly.

(4.11) PROPOSITION: *Suppose that* X *is a metric space,* $X = \bigcup X_\alpha$*, where the* X_α *have asymptotic dimension* $\leqslant n$ *uniformly. Suppose also that for each* $s > 0$ *there is a* $Y_s \in X$*, of asymptotic dimension* $\leqslant n$*, such that the sets* $X_\alpha \setminus Y_s$ *(fixed* α*, varying* s*) are* s*-separated. Then* X *has asymptotic dimension* $\leqslant n$*.*

The condition with the Y_s is a 'coarse excision' condition, similar to that used in the coarse Mayer-Vietoris principle. The proof of the proposition is analogous to the proof of the earlier one.

(4.12) LEMMA: *Let* X *and* Y *be metric spaces and let* $\pi\colon X \to Y$ *be a Lipschitz map. Suppose that* Y *has asymptotic dimension* $\leqslant n$ *and that, for each fixed* $R > 0$*, the inverse images* $\pi^{-1}(B(y;R)) \subseteq X$ *have asymptotic dimension* $\leqslant k$*, uniformly in* y*. Then* X *has finite asymptotic dimension; in fact,* $\operatorname{asdim} X \leqslant (n+1)(k+1) - 1$*.*

The inequality is not sharp — the correct estimate is $\operatorname{asdim} X \leqslant n + k$, but for this one needs simplicial techniques.

PROOF: Let $r > 0$ be given. Suppose that the map π has Lipschitz constant λ. Then there is an $R > 0$ for which there exist $(n+1)$ λr-disjoint, R-bounded families $\mathcal{U}_0, \dots, \mathcal{U}_n$ that together cover Y. Consider the inverse image of one of these familes, say

$$\pi^{-1}(\mathcal{U}_i) := \{\pi^{-1}(U) : U \in \mathcal{U}_i\}.$$

This is an r-disjoint family of sets which uniformly have asymptotic dimension $\leqslant k$. Therefore, there is an $S > 0$ such that the union $\bigcup \pi^{-1}(\mathcal{U}_i)$ can be covered by $(k+1)$ r-disjoint, S-bounded families. Performing this construction for each i we cover the whole of X by $(n+1)(k+1)$ r-disjoint, S-bounded families, as required. ■

(4.13) REMARK: One can sharpen the argument somewhat, in a way that will be important later. Let τ be the type function associated to the estimate $\operatorname{asdim} Y \leqslant n$, and let λ be the Lipschitz constant of π. Suppose that, for each $r > 0$, the inverse images $\pi^{-1}(B(y;\tau(\lambda r)))$ have asymptotic dimension $\leqslant k$ 'on scale r', uniformly in y; I mean that there is an $S > 0$ such that each such inverse image can be written as the union of $(k+1)$ r-disjoint, S-bounded families. Then the conclusion holds (with the same argument).

This principle can be usefully applied when $Y = \mathbb{R}$ and π is a 1-Lipschitz map, for instance the distance from a fixed point. Observe that $\operatorname{asdim} \mathbb{R} \leqslant 1$ with type function $\tau(r) = r$. We see that it is enough to prove (with appropriate uniformity) that for each r, the inverse images under π of intervals of length r have finite asymptotic dimension on scale r.

(4.14) COROLLARY: *Suppose that a (finitely generated) group* G *acts by isometries on a metric space* X*. Suppose that*

(a) X *has finite asymptotic dimension;*

(b) The 'pro-stabilizers' have finite asymptotic dimension; by this I mean that there should exist a point $x_0 \in X$ *(the condition then holds for all points) and a number* k *such that for each* $R > 0$ *the subset*

$$\{g \in G : d(gx_0, x_0) \leqslant R\}$$

has finite asymptotic dimension $\leqslant k$*.*

Then G *has finite asymptotic dimension.*

PROOF: Without loss of generality replace X by the orbit of x_0. Then apply lemma 4.12 to the map $\pi\colon G \to X$, $g \mapsto gx_0$. ■

(4.15) COROLLARY: *Let*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & G \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

be an exact sequence of finitely generated groups. If K *and* H *have finite asymptotic dimension, then so does* G*.*

PROOF: Let G act on H by left translation. We metrize both G and H using the same finite generating set, say $g_1, \ldots, g_k$. To apply Corollary 4.14, we need only check that the pro-stabilizers of the identity are just the metric neighborhoods of K in G, and therefore are coarsely equivalent to K. That is, we need to check that if $g \in G$ is such that $\pi(g)$ can be written as a word of length r in $\pi(g_1), \ldots, \pi(g_k)$, then g can be written as the product of an element of K and a word of length r in $g_1, \ldots, g_k$. But this is obvious. ■

In the opposite direction, it is certainly true that any (finitely generated) subgroup of a group of finite asymptotic dimension must itself have finite asymptotic dimenion. Since the inclusion of a finitely generated subgroup is a coarse equivalence onto its image, this is included in our results above. However, it is not the case that finite asymptotic dimension is preserved under quotients. Since free groups have finite asymptotic dimension, this will be shown as soon as we exhibit a single example of a finitely generated group which does *not* have finite asymptotic dimension.

Our example is the 'wreath product' of $\mathbb{Z}$ with $\mathbb{Z}$. To construct this group, let H be the set of finitely supported maps $\mathbb{Z} \to \mathbb{Z}$, and let u and v be the permutations of H defined by

$$uf(n) = f(n) + \delta_{n0}, \quad vf(n) = f(n+1).$$

The group G generated by u and v is called the *restricted wreath product* of $\mathbb{Z}$ by $\mathbb{Z}$. Now for any n, one easily sees that the elements

$$u, v^{-1}uv, \ldots, v^{-(n-1)}uv^{(n-1)}$$

generate a subgroup of G isomorphic to $\mathbb{Z}^n$. Consequently, $\operatorname{asdim} G \geqslant n$, and since n is arbitrary, G cannot have finite asymptotic dimension.

(4.16) Corollary: *The hyperbolic plane $\mathbb{H}^2$ has finite asymptotic dimension. More generally, the hyperbolic space $\mathcal{H}(X)$ built on a geodesic metric space X (see definition 1.21) has finite asymptotic dimension if X has finite asymptotic dimension of linear type.*

Proof: Recall that $\mathcal{H}(X)$ is the 'space of balls in X'. In other words it is equal (as a set) to $X \times \mathbb{R}^+$. We can define its coarse structure by a metric as follows: for points $z = (x,t)$ and $z' = (x', t')$ in $\mathcal{H}(X)$, put

$$\rho(z, z') = 2 \log \left(\frac{d(x,x') + \max\{t,t'\}}{\sqrt{tt'}} \right).$$

This is a metric: let us check the triangle inequality for three points z, z', z'': it amounts to

$$t'\big(d(x,x'') + \max\{t,t''\}\big) \leqslant \big(d(x,x') + \max\{t,t'\}\big)\big(d(x',x'') + \max\{t',t''\}\big).$$

This follows from

$$t' \max\{t,t''\} \leqslant \max\{t,t'\} \max\{t',t''\}$$

and

$$t'd(x,x'') \leqslant t'\big(d(x,x') + d(x',x'')\big) \leqslant \max\{t,t'\}d(x',x'') + \max\{t',t''\}d(x,x').$$

The metric ρ defines the coarse structure described earlier on $\mathcal{H}(X)$.

We are going to apply Lemma 4.12 to the 1-Lipschitz map $\mathcal{H}(X) \to \mathbb{R}$ defined by $\pi(x,t) = \log t$. We need therefore to show that for each fixed $\lambda > 1$ the strips

$$S_{\lambda,a} = \{(x,t) : a \leqslant t \leqslant a\lambda\}$$

have finite asymptotic dimension, uniformly in a. But if z and z' belong to the strip $S_{\lambda,a}$ then we have by simple estimates

$$2\log\left(\frac{d(x,x')}{\lambda a} + 1\right) \leqslant d(z,z') \leqslant 2\log\left(\frac{d(x,x')}{a} + \sqrt{\lambda}\right).$$

Thus the map $(x,t) \mapsto x$ is a coarse equivalence from $S_{\lambda,a}$ to $\frac{1}{a}X$, uniformly in a. By linear type the spaces $\frac{1}{a}X$ have finite asymptotic dimension, uniformly in a, and it follows that the strips $S_{\lambda,a}$ have finite asymptotic dimension, uniformly in a, also. ■

4.3 Free products; groups acting on trees

(4.17) THEOREM: *Let* G *be a finitely generated group which acts cocompactly on a tree* T *such that all stabilizers of vertices are finitely generated and have finite asymptotic dimension. Then* G *has finite asymptotic dimension.*

(Bass-Serre theory: free products, amalgamated products, HNN extensions).

PROOF: Granted Corollary 4.14, all we have to do is to show that the pro-stabilizers have finite asymptotic dimension. Suppose then that for all $x \in T$, $\operatorname{asdim} G_x \leqslant n$ (the existence of such an n follows from our hypothesis and from cocompactness).

■

4.4 Hyperbolic groups

Our objective in this section is to prove that every finitely generated hyperbolic group (in the sense of Gromov) has finite asymptotic dimension. Our proof applies to a somewhat wider category of spaces too. The most general results follow from the work of Bonk–Schramm (*Embeddings of Gromov hyperbolic spaces*); they prove that any hyperbolic metric space that has *uniform growth at some scale* 'almost embeds' into real hyperbolic space $\mathbb{H}^n$; finite asymptotic dimension certainly follows from this.

Recall that a metric space X is δ-*hyperbolic* if for all triples of points $x, y, z) \in X$ we have

$$(x|z) \leqslant \min\{(x|y), (y|z)\} - \delta,$$

where all Gromov products are taken with respect to a fixed base point.

(4.18) Definition: *A uniformly discrete metric space* X *has* uniform growth *(at all scales) if, for every* $a > 0$*, there is a constant* C *such that, for every* $R > 0$*, each ball of radius* $R + a$ *in* X *can be covered by at most* C *balls of radius* R.

This is a rather weak condition. For instance, every discrete group satisfies it because in a group

$$B(e; R + a) = \bigcup_{x \in B(e;a)} B(e; R)$$

and so we may take C to be the cardinality of $B(e; a)$. More generally any space that is of bounded geometry and aprooximatley geodesic has uniform growth.

(4.19) Theorem: *A hyperbolic group has finite asymptotic dimension.*

Proof: With minor modifications our proof will apply to all uniformly discrete hyperbolic spaces of uniform growth. We are going to apply the fibration lemma (Lemma 4.12, in the strengthened form of Remark 4.13) to the 1-Lipschitz map $\pi\colon X \to \mathbb{R}^+$ given by

$$\pi(x) = |x| = d(x, e).$$

Thus, what we need to show is that there exists a C such that, for each fixed r, there is an $S > 0$ such that the shells

$$S_k = \{x : kr \leqslant |x| \leqslant (k+1)r\}$$

can each be covered by C r-disjoint, S-bounded families. Without loss of generality we assume that r and k are integers.

Let $\{x_i : i = 1, \ldots, N\}$ be a subset of S_k such that

(a) $|x_i| = kr$ for all i,

(b) $(x_i|x_j) \leqslant (k - \frac{1}{2})r$,

and which is maximal with respect to these properties. (Compare the proof of Proposition 4.5.) Define

$$U_i = \{x \in S_k : (x|x_i) \geqslant (k - \tfrac{1}{2})r - \delta\}.$$

We shall show

(i) The sets U_i cover S_k,

(ii) Each U_i has diameter at most $3r + 2\delta$,

(iii) There is an absolute constant C such that no more than C of the sets U_i can meet any given ball of radius r.

This will give us the desired conclusion.

Proof of (i) To show that the U_i cover S_k, let $x \in S_k$, and let x' be a point on a geodesic from x' to the origin and having $|x'| = kr$. Thus $(x|x') = kr = |x'|$. By maximality, there is some x_i with $(x'|x_i) \geqslant (k - \frac{1}{2})r$. Then

$$(x|x_i) \geqslant \min\{(x|x'), (x'|x_i)\} - \delta \geqslant (k - \tfrac{1}{2})r - \delta,$$

and so $x \in U_i$.

Proof of (ii) Suppose that $x, y \in U_i$. Then

$$(x|y) \geqslant \min\{(x|x_i), (x_i, y)\} - \delta \geqslant (k - \tfrac{1}{2})r - \delta.$$

Thus

$$d(x, y) = |x| + |y| - 2(x|y) \leqslant 2(k+1)r - 2((k - \tfrac{1}{2})r - \delta) = 3r + 2\delta$$

as required.

Proof of (iii) Suppose that U_i meets $B(x, r)$, so that there is $y \in S_k$ with $d(x, y) \leqslant r$ and $(y|x_i) \geqslant (k - \frac{1}{2})r - \delta$. Let x'' be a point on a geodesic from e to x with $|x''| = (x|x'') = \lfloor (k - \frac{1}{2})r \rfloor$, and similarly let y'' be a point on a geodesic from e to y with $|y''| = (y|y'') = \lfloor (k - \frac{1}{2})r \rfloor$. Then

$$(y''|x_i) \geqslant \min\{(y''|y), (y|x_i)\} - \delta \geqslant (k - \tfrac{1}{2})r - 2\delta - 1.$$

Consequently

$$d(y'', x_i) = |y''| + |x_i| - 2(y''|x_i) \leqslant \tfrac{1}{2}r + 4\delta + 2.$$

Moreover, since $|x|, |y| \geqslant kr$ while $d(x,y) \leqslant r$, we have $(x|y) \geqslant (k - \frac{1}{2})r$, and therefore

$$(x''|y'') \geqslant \min\{(x''|x), (x|y), (y|y'')\} - 2\delta \geqslant (k - \tfrac{1}{2})r - 2\delta - 1.$$

Hence

$$d(x'', y'') = |x''| + |y''| - 2(x''|y'') \leqslant 4\delta + 2.$$

We conclude that if $B(x;r)$ meets $\mathcal{U}_i$, then $x_i \in B(x''; \frac{1}{2}r + 8\delta + 4)$. On the other hand since $(x_i|x_j) \leqslant (k - \frac{1}{2})r$, $|x_i| = |x_j = kr$, we easily conclude that $d(x_i, x_j) \geqslant r$ if $i \neq j$. Hence the number of sets U_i that contain x is bounded by the maximum cardinality of an r-separated subset of a ball of radius $\frac{1}{2}r + 8\delta + 4$. But this in turn is bounded by the minimum number of $\frac{1}{2}r$-balls needed to cover such a subset, and uniform growth tells us that this number is bounded by an absolute constant C. We are done. ■

4.5 Analytic implications

(This subsection is hacked about from a paper we are writing)

Recall that the coarse Baum–Connes conjecture for a metric space Z provides a conjectural computation for the K-theory of the 'Roe algebra' $C^*(Z)$, which is the C^*-algebra generated by the locally compact, finite propagation operators on a Hilbert space H equipped with a representation of $C_0(Z)$. The choice of H is of minor importance, but we will need to specify it at some points below; let us agree to choose a measure μ on Z, and then to set $H = L^2(Z, \mu) \otimes \ell^2(\mathbb{Z})$. Where necessary we will use the notation $C^*(Z, \mu)$ to specify this particular choice of H.

(4.20) Remark: It will be important in the ensuing argument to keep clearly in mind the distinction between the dense subalgebra of $C^*(Z)$ consisting of finite propagation operators, and the whole algebra $C^*(Z)$ which consists of *norm limits* of such operators.

It is easy to see that finite propagation operators on H (not necessarily locally compact) act as multipliers on $C^*(Z)$. Among such operators are those induced by translations of Z. Hence:

(4.21) LEMMA: *Let Z be a coarse space and let Γ be a group of translations of Z. Let μ be a measure on Z invariant under Γ. Then the unitary representation of Γ on $L^2(Z, \mu)$ induced by the action of Γ on Z extends to a $*$-homomorphism*

$$\pi \colon C^*_{\max}(\Gamma) \to M(C^*(Z, \mu)).$$

PROOF: The hypothesis that Γ acts by translations says exactly that the unitaries $\pi(\gamma)$, $\gamma \in \Gamma$, have finite propagation. ■

We now proceed to the construction of a counterexample to the coarse Baum–Connes conjecture.

(4.22) CONSTRUCTION: *Let Γ be a finitely generated discrete group such that:*

- *(a) Γ is residually finite; there is an increasing sequence $\{\Gamma_n\}$ of finite index normal subgroups of Γ such that $\bigcap \Gamma_n = \{1\}$;*
- *(b) Γ has Kazhdan's property T [?];*
- *(c) Γ has finite asymptotic dimension [?, ?, ?].*

For example, conditions (a)–(c) are satisfied by the group $SL(3, \mathbb{Z})$.

Define a space X to be a coarse disjoint union $\bigsqcup_n \Gamma/\Gamma_n$; we use a fixed set of generators for Γ to define the right-invariant word metrics on all the Γ/Γ_n, and we arrange the metric on the disjoint union so that the distances between the various 'components' $X_n = \Gamma/\Gamma_n$ of X are very rapidly increasing.

It is this space X that will turn out to be a counterexample to the coarse Baum–Connes conjecture. Notice that Γ acts on X by translations (on the left).

(4.23) CONSTRUCTION: *With X the space of Construction 4.22, we will define a certain projection in $C^*(X)$. We build $C^*(X)$ as an algebra of operators on the Hilbert space $H = \ell^2(X) \otimes \ell^2(Z)$. Let e_1 be a fixed rank one projection on $\ell^2(\mathbb{Z})$, and let $p_1 = 1 \otimes e_1$; it is a projection in $C^*(X)$. Let e_2 be the Kazhdan projection in $C^*_{\max}(\Gamma)$, and let $p_2 = \pi(e_2)$ be the projection in $M(C^*(X))$ obtained by the construction of Lemma 4.21. (One easily sees that p_2 is the projection onto those elements of H that are constant on each 'component' X_n of X; in particular p_2 is not a* *finite propagation operator.) Since* p_1 *is invariant under the action of* Γ, $p_1p_2 = p_2p_1$ *is a projection, call it* p, *in* $C^*(X)$. *It will turn out that the class* $[p] \in K_0(C^*(X))$ *does not belong to the image of the assembly map, thus showing that this map is not surjective.*

As in Example ??, we are going to construct a 'lower' and a 'higher' index (dimension) function on $K_0(C^*(X))$. These indices will disagree on $[p]$, and we will conclude that $[p]$ does not belong to the image of the assembly map.

At several points we will make use of the following well-known functional calculus lemma:

(4.24) LEMMA: *Let* x *be a self-adjoint element of a unital* C^**-algebra* A. *If* $\|x^2 - x\| < \frac{1}{4}$, *then there is a projection* $q \in A$ *such that* $\|x - q\| < \frac{1}{2}$. *Moreover, any two such projections are unitarily equivalent.* □

(4.25) CONSTRUCTION: *Here is the construction of the 'lower index', corresponding to* φ *in the argument of Example ??. Let* X *be as in Construction 4.22; all we need here in fact is that* X *is a coarse disjoint union of compact 'components'* X_n. *Let* $q \in C^*(X)$ *be a projection*[15] *representing* $[q] \in K_0(C^*(X))$. *We may truncate* q *to each of the components* X_n *of* X, *thus obtaining a sequence* x_n *of compact operators. One sees easily from the definition of* $C^*(X)$ *that for all sufficiently large* n *the truncations* x_n *satisfy* $\|x_n^2 - x_n\| < \frac{1}{4}$. *By the lemma above, for each such* n *there is a finite-rank projection* q_n *such that* $\|x_n - q_n\| < \frac{1}{2}$, *and that the rank* $\dim \operatorname{Im} q_n$ *is determined by* q. *The assignment* $[q] \mapsto \{\dim \operatorname{Im} q_n : n \in \mathbb{N}\}$ *(defined for sufficiently large* n*) gives a map*

$$\varphi\colon K_0(C^*(X)) \to \prod_{n=1}^{\infty} \mathbb{Z} \Big/ \bigoplus_{n=1}^{\infty} \mathbb{Z}.$$

(4.26) REMARK: It will be helpful to give another (equivalent) description of the map φ. By restricting an operator $T \in C^*(X)$ to each of the components X_n of X, we obtain a $*$-homomorphism

$$\beta\colon C^*(X) \to \prod_{n=1}^{\infty} \mathfrak{K}(H_n) \Big/ \bigoplus_{n=1}^{\infty} \mathfrak{K}(H_n),$$

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[15] As observed in [?], $C^*(X)$ is stable and has an approximate unit of (finite-propagation) projections, so that every K_0-class for $C^*(X)$ can be described by a formal difference of projections in $C^*(X)$.

where H_n is the subspace of H supported on X_n. One sees easily that the K_0-theory of the 'corona' algebra appearing on the right is $\prod_{n=1}^{\infty} \mathbb{Z} / \bigoplus_{n=1}^{\infty} \mathbb{Z}$, and φ is just the map induced on K-theory by β. For future reference let us recall that the identification $K_0(\mathfrak{K}) \to \mathbb{Z}$ used here is induced by the usual (semifinite, semicontinuous) trace on $\mathfrak{K}$.

(4.27) CONSTRUCTION: *Let notation be as in Construction 4.22. In defining our 'higher index' we are going to make use of another auxiliary 'corona' C^*-algebra A. Recall the notation $|\Gamma|$ for the underlying metric space of Γ. Let $A_n = C^*_{\Gamma_n}(|\Gamma|)$ denote the algebra of operators on $\ell^2(|\Gamma|) \otimes \ell^2(\mathbb{Z})$ generated by the finite propagation operators that are invariant under conjugation by the right-translation action of the subgroup Γ_n, and let A denote the direct product C^*-algebra $\prod_{n=1}^{\infty} A_n$. Let I denote the C^*-algebra direct sum $\bigoplus_{n=1}^{\infty} A_n$; it is an ideal in A (consisting of those sequences in the product whose norms tend to zero with n). Our auxiliary algebra is the quotient C^*-algebra A/I.*

The algebra A_n is Morita equivalent to $C^*_r(\Gamma_n)$. As such it acquires a 'von Neumann trace' τ_n, corresponding to the canonical trace on $C^*_r(\Gamma_n)$. Explicitly, this may be defined on an operator $S \otimes T$ (an elementary tensor acting on $\ell^2(|\Gamma|) \otimes \ell^2(\mathbb{Z})$) by

$$\tau_n(S \otimes T) = \mathrm{Tr}(T) \sum_{\gamma \in \Gamma/\Gamma_n} \langle S e_\gamma, e_\gamma \rangle,$$

where the sum is extended over a single representative γ for each coset in Γ/Γ_n and e_γ is the corresponding basis vector in $\ell^2(\Gamma)$. Each von Neumann trace is semifinite and semicontinuous and therefore[16] *defines a dimension function $K_0(A_n) \to \mathbb{R}$. Putting together these various von Neumann traces one may define a 'higher index function'*

$$\tau \colon K_0(A/I) \to \prod_{n=1}^{\infty} \mathbb{R} \Big/ \bigoplus_{n=1}^{\infty} \mathbb{R}.$$

To define τ on a projection q in the quotient, lift to an element $x = \{x_n\}$ in A and note that all but finitely many of the x_n satisfy $\|x_n^2 - x_n\| < \frac{1}{4}$. Thus, by Lemma 4.24, all but finitely many x_n satisfy $\|x_n - q_n\| < \frac{1}{2}$ for some projection q_n in A_n, unique up to unitary equivalence. The list $\{\tau_n(q_n)\}$ (n sufficiently large) then provides the desired higher index function.

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[16] See [?].

(4.28) CONSTRUCTION: *Let notation be as in Constructions 4.22 and 4.27. There is a 'transfer' homomorphism* $\alpha\colon C^*(X) \to A/I$ *which will be crucial to the proof. To define it, first suppose that* $T \in C^*(X)$ *is of finite propagation* r *say. For sufficiently large* n*, the ball in* Γ *of radius* $2r$ *about the identity does not meet the subgroup* Γ_n *in any non-identity element. For such an* n *let us define a* Γ_n*-periodic finite propagation operator* $\alpha_n(T)$ *on* Γ *by*

$$\langle \alpha_n(T)e_\gamma, e_{\gamma'}\rangle = \begin{cases} \langle Te_{\pi_n(\gamma)}, e_{\pi_n(\gamma')}\rangle & (d(\gamma,\gamma') \leqslant r) \\ 0 & (d(\gamma,\gamma') > r) \end{cases}.$$

It is then easy to see that the assignment $T \mapsto \{\alpha_n(T)\}$ *gives a* $*$*-homomorphism* α *from the dense subalgebra* $FP(X)$ *of* $C^*(X)$ *made up of finite propagation operators to the quotient* C^**-algebra* A/I.

We are going to use finite asymptotic dimension to show that the homomorphism $\alpha\colon FP(X) \to A/I$ of Construction 4.28 extends continuously to a $*$-homomorphism from $C^*(X)$ to A/I. Recall [?, ?, ?] that a metric space X has asymptotic dimension $\leqslant d$ if the following is true: for each $r > 0$ there is an $s > 0$ such that one can partition X into the union of $(d+1)$ (r,s)-*blocks*. By definition, an (r,s)-block is a disjoint union of pieces each of which individually has diameter $\leqslant s$ and which are separated by at least r.

(4.29) LEMMA: *Suppose that* X *is a uniformly discrete bounded geometry space having finite asymptotic dimension* $\leqslant d$*. For each* r *there is an* s *such that the following is true: Let* T *be an operator on* $\ell^2(X)$ *with propagation* $\leqslant r$*. Then*

$$\|T\| \leqslant (d+1)\sup\{\|T\xi\|_2 : \|\xi\| = 1, \operatorname{diam}\operatorname{Supp}(\xi) \leqslant s\}.$$

PROOF: Decompose X into blocks $X_0, \dots, X_d$ associated to scale $2r$. Let P_i be the orthogonal projection given by multiplication by the characteristic function of the i'th block. Then

$$T = \sum_{j=0}^{d} TP_j.$$

There are $(d+1)$ terms here. Let us estimate the norm of each one. Write the block X_j as the disjoint union of pieces K_i each of which has diameter at most s and such that $d(K_i, K_{i'}) \geqslant 2r$ if $i \neq i'$. Let Q_i denote the orthogonal projection given by multiplication by the characteristic function of K_i, and notice that the ranges of TQ_i and $TQ_{i'}$ are orthogonal if $i \neq i'$. Writing $C = \sup\{\|T\xi\|_2 : \|\xi\| = 1, \operatorname{diam}\operatorname{Supp}(\xi) \leqslant s\}$ for the constant appearing in the statement of the proposition, we therefore have

$$\|TP_j\xi\|^2 = \sum_i \|TQ_i\xi\|^2 \leqslant C^2 \sum_i \|Q_i\xi\|^2 \leqslant C^2\|\xi\|^2$$

and therefore $\|T\| \leqslant C$ as asserted. ■

(4.30) Lemma: *Let notation be as in Constructions 4.22, 4.27, and 4.28. The homomorphism α described above extends continuously to a $*$-homomorphism $C^*(X) \to A/I$.*

The crucial fact here is that $|\Gamma|$ has finite asymptotic dimension. Note that it is *not true* in general that the space X will have finite asymptotic dimension.

Proof: We will show that there is a constant c having the following property: for any finite propagation operator T, one has

$$\|\alpha(T)\| \leqslant c\|T\|$$

where the norms are operator norms in both cases. Notice that the disjoint union of countably many copies of $|\Gamma|$ is also a space of finite asymptotic dimension, say d; and $\alpha(T)$ can be considered as an operator of propagation $\leqslant r$ defined on this space. According to Lemma 4.29 there is a constant s such that

$$\|\alpha(T)\| \leqslant (d+1)\sup\{\|\alpha(T)\xi\|_2 : \|\xi\| = 1, \operatorname{diam}\operatorname{Supp}(\xi) \leqslant s\}.$$

Now, for all but finitely many n, the ball in Γ of radius $2(r+s)$ does not meet any non-identity element of Γ_n. Suppose n such and let ξ be supported in a ball $B(x; s)$ in Γ. The projection $\pi_n : \Gamma \to \Gamma_n$ is bijective on $B(x; s+r)$; this bijection gives a unitary isomorphism U from the ℓ^2 functions on the ball $B(x; s+r)$ in Γ to the ℓ^2 functions on the image ball $B(\pi_n(x); s+r)$ in Γ/Γ_n. Moreover, this unitary isomorphism satisfies

$$\alpha(T)\xi = U^*TU\xi.$$

We conclude that for all but finitely many n the right-hand side of the displayed inequality above is bounded by $\|T\|$, and the result follows. ■

Lemma 4.30 allows us to define our 'higher index' function

$$\tau \circ \alpha_* \colon K_0(C^*(X)) \to \prod_{n=1}^{\infty} \mathbb{R} \Big/ \bigoplus_{n=1}^{\infty} \mathbb{R},$$

by composing the map on K-theory induced by $\alpha\colon C^*(X) \to A/I$ with the dimension function τ of Construction 4.27. We now observe that the higher and lower indices agree on the image of the assembly map.

(4.31) LEMMA: *Let Z be a proper bounded geometry metric space. Any K-theory class belonging to the image of the coarse assembly map $KX_0(Z) \to K_0(C^*(Z))$ can be represented by a formal difference of finite-propagation idempotents over (the unitalization of) $C^*(Z)$.*

In our terminology, an *idempotent* is equal to its square, but need not be self-adjoint. We do not know whether the lemma is true when *idempotent* is replaced by *projection*.

PROOF: We refer to [?] for the construction of the coarse assembly map. One builds a 'coarsening sequence' or 'anti-Čech sequence' $\{Z_n\}$ from the space Z, associated to which are maps

$$Z \to Z_1 \to Z_2 \to \dots;$$

each map appearing here is a coarse equivalence. By definition, the coarse homology $KX_0(Z)$ appearing on the left of the assembly map is the direct limit $\varinjlim K_0(Z_n)$. It follows that each coarse K-homology class comes from the K-homology of some Z_n. Its image under assembly comes from composing the coarse index map $K_0(Z_n) \to K_0(C^*(Z_n))$ with the isomorphism induced by coarse equivalence $K_0(C^*(Z_n)) \to K_0(C^*(Z))$. By virtue of its construction the latter isomorphism preserves finite propagation. It suffices then to show that the coarse index of any K-homology class has a finite propagation representative. Any K-homology class itself has a finite propagation representative [?, Lemma 6.2], say a finite propagation pseudolocal operator U such that $UU^* - 1$ and $U^*U - 1$ are locally compact. The coarse index is a boundary map in K-theory, and so it is represented by the well-known formula

$$[WPW^{-1}] - [P], \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$W = \begin{bmatrix} (2-UU^*)U & UU^*-1 \\ 1-U^*U & U^* \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} U^* & 1-U^*U \\ UU^*-1 & (2-UU^*)U \end{bmatrix}.$$

Here W and W^{-1}, and therefore WPW^{-1}, are finite propagation operators. The idempotents P and WPW^{-1} ■

(4.32) PROPOSITION: *Suppose that* $[q] \in K_0(C^*(X))$ *is represented by an idempotent of finite propagation. Then* $\varphi[q] = \tau(\alpha_*[q]) \in \prod_{n=1}^{\infty} \mathbb{R} / \bigoplus_{n=1}^{\infty} \mathbb{R}$.

PROOF: Because both φ and τ are defined in terms of the maps on K-theory induced by semifinite, semicontinuous traces, it will suffice to show that if T is an operator of finite propagation, then for sufficiently large n we have

$$\mathrm{Tr}(T_{|H_n}) = \tau_n(\alpha_n(T)).$$

This is a simple computation:

$$\widetilde{\tau}_n(\alpha_n(T)) = \sum_{\gamma \in \Gamma/\Gamma_n} \langle Te_\gamma, e_\gamma \rangle = \mathrm{Tr}\, T,$$

as required. ■

(4.33) THEOREM: *Let* $p \in C^*(X)$ *be the projection of Construction 4.23. The class* $[p] \in K_0(C^*(X))$ *does not belong to the image of the coarse assembly map. Consequently, the Coarse Baum–Connes conjecture does not hold for the space* X *of Construction 4.22.*

PROOF: Plainly p has rank one on each component of X and thus $\varphi[p] = (1, 1, 1, \dots)$. Let us calculate the transfer $\alpha(p) \in A/I$. Let $x \in \mathbb{C}[\Gamma]$ be a finite linear combination of group elements which approximates the Kazhdan projection e_2 (in the $C^*_{\max}$ norm) to within $\frac{1}{2}$. Let $T = \pi(x)$ be the finite propagation operator on H obtained from x; then $\|T - p\| \leqslant \frac{1}{2}$. The lift $\alpha(T)$ is defined and, because α extends continuously to a C^*-homomorphism, $\|\alpha(T) - \alpha(e)\| < \frac{1}{2}$. On the other hand, we may describe $\alpha(T)$ explicitly; it is simply the 'inflation' (product of infinitely many copies) of the operator on $\ell^2(\Gamma)$ obtained from x via the regular representation. As such it lies within $\frac{1}{2}$ of the operator on $\ell^2(X)$ obtained from the Kazhdan projection e_2 via the regular representation, and the latter operator is zero (there are no fixed vectors in the regular representation). We conclude that $\|\alpha(p)\| < 1$ and since $\alpha(p)$ is a projection, it must be 0. Our conclusion is that $\tau(\alpha_*[p]) = (0, 0, 0, \dots)$. It follows from Lemma 4.31 and Proposition 4.32 that the K-theory class $[p]$ does not belong to the image of the assembly map. ■

4.6 Dranishnikov's Embedding Theorem

Let us take stock of the finite asymptotic dimensional spaces that we have manufactured so far. Every tree has finite asymptotic dimension. Moreover, the property of having finite asymptotic dimension is stable under products, and under taking subspaces, so that any subspace of a finite product of trees certainly has finite asymptotic dimension.

The remarkable embedding theorem of Dranishnikov (reference) asserts that *all* spaces of finite asymptotic dimension are of this kind.

(4.34) THEOREM: *Let* X *be a geodesic space of bounded geometry and having* $\operatorname{asdim}(X) \leqslant n$. *Then* X *is coarsely embeddable in a product of* $(n+1)$ *locally finite trees.*

Begin by reviewing properties of 1-Lipschitz maps
1-Lipschitz extension theorem
1-Lipschitz gluing theorem

(4.35) LEMMA: *Good anti-Cech system*

proof of theorem.

5 Coarse Embedding

5.1 Basic properties

5.2 Negative type functions and embedding in Hilbert space

5.3 Stable Banach spaces and coarse embedding

5.4 Property T and coarse embedding

5.5 Measure equivalence versus coarse equivalence

6 Groupoids and coarse geometry

6.1 Basic facts about groupoids

6.2 Decomposable sets and $\beta(X \times Y)$

Recall from our earlier discussion of the Stone-Cech compactification βX of a (discrete) space X that clopen subsets of βX correspond to (more precisely: are the closures of) *arbitrary* subsets of X. Moreover, the clopen sets of βX form a *basis* for its topology.

We will want to investigate the clopen subsets of a *product* $\beta Y \times \beta Z$ of Stone-Cech compactifications. Again, we will find that all such sets are the closures of certain subsets of $Y \times Z$. But in contrast to the earlier case, not all subsets of $Y \times Z$ may arise.

(6.1) DEFINITION: *Let Y and Z be sets. A subset of $Y \times Z$ is* decomposable *if it is a finite union $\bigcup_{i=1}^{n} A_i \times B_i$ of products of subsets $A_i \subseteq Y$ and $B_i \subseteq Z$.*

Note that, increasing n if necessary, we may always assume that the union is disjoint.

(6.2) PROPOSITION: *The clopen subsets of $\beta Y \times \beta Z$ are precisely the closures of the decomposable subsets of $Y \times Z$.*

PROOF: If $A \subseteq Y$ and $B \subseteq Z$, clearly $\overline{A \times B} = \overline{A} \times \overline{B}$ is clopen. Thus the closure of any decomposable set is clopen.

Suppose that $W \subseteq \beta Y \times \beta Z$ is clopen. Since W is open and the topologies on βY and βZ have a basis of clopen sets, for each $w \in W$ there exist clopen $U_w \subseteq \beta Y$ and $V_w \subseteq \beta Z$, having $w \in (U_w \times V_w) \subseteq W$. Since W is closed and therefore compact, finitely many of these products cover W. Thus

$$W = \bigcup_{i=1}^{n} U_{w_i} \times V_{w_i}$$

for some $\{w_i\}$. Writing each U_w as the closure of a subset $A_w \subseteq Y$ and each V_w as the closure of a subset $B_w \subseteq Z$, we see that $W = \overline{\bigcup_{i=1}^{n} A_{w_i} \times B_{w_i}}$ is the closure of a decomposable set. ■

More generally, suppose that $L \subseteq Y \times Z$ is given. We'll say that a subset of L is *relatively decomposable* if it is of the form $L \cap S$, where S is decomposable. Then we have

(6.3) PROPOSITION: *Let* $K \subset \beta Y \times \beta Z$ *be closed. Then the clopen subsets of* K *(in its relative topology) are precisely those of the form* $\overline{S} \cap K$*, where* S *is decomposable. In particular, if* $L = K \cap (X \times Y)$*, then the intersection with* L *of any clopen subset of* K *is relatively decomposable.*

PROOF: One direction is obvious; it follows from the previous proposition that the sets described are relatively clopen. For the converse, we must prove that any clopen subset of K can be written as the intersection of K with a clopen subset of $\beta X \times \beta Y$. To do this, repeat the argument of the previous proposition. ■

(6.4) PROPOSITION: *Let* $E \subseteq X \times Y$ *and let* $\varphi \colon \overline{E} \to \beta X \times \beta Y$ *be the map obtained by continuously extending* $(s, t) \colon E \to X \times Y$*. Then* φ *is injective if and only if every subset of* E *is relatively decomposable.*

In particular the universal map $\beta(X \times Y) \to \beta X \times \beta Y$ *is not injective unless one or both of* X, Y *are finite.*

PROOF: If φ is injective it is a homeomorphism onto its image (since it is a continuous map from a compact space to a Hausdorff space).

The closure (in $\overline{E}$) of each subset of E is clopen; indeed $\overline{E}$ is homeomorphic to βE. Therefore, if φ is injective, the closure of every subset of $\varphi(E)$ is clopen in $\varphi(\overline{E})$. It follows from the previous proposition that every subset of E is relatively decomposable.

Conversely, suppose that every subset of E is relatively decomposable. Consider the map induced by φ on function algebras

$$\varphi^* \colon \ell^\infty(X) \otimes \ell^\infty(Y) \to \ell^\infty(E).$$

We must show that φ^* is injective, and since it is a C^*-homomorphism, it suffices to show that its range is dense. But, given $\varepsilon > 0$, any $f \in \ell^\infty(E)$ can be approximated within ε by a finite linear combination of characteristic functions of subsets of E. These subsets are relatively decomposable; which means that their characteristic functions are restrictions to E of elements of the tensor product $\ell^\infty(X) \otimes \ell^\infty(Y)$. We conclude that φ^* has dense range, as required. ■

We will now use this discussion to show that there is no 'natural' topological groupoid structure on $\beta(X \times X)$.

(6.5) PROPOSITION: *Let* X *be a (discrete) infinite space. Then the multiplication map of the pair groupoid* $X \times X$ *does* not *extend to a* *continuous groupoid multiplication on* $\beta(X\times X)$*, equipped with the source and target maps* $s, t\colon \beta(X\times X) \to \beta X$ *that continuously extend those of the pair groupoid.*

PROOF: Suppose that the desired multiplication did exist. It would be a continuous map $m\colon K \to \beta(X \times X)$, where

$$K = \beta(X\times X) \times_{\beta X} \beta(X\times X) = \{(\xi, \eta) \in \beta(X\times X) \times \beta(X\times X) : t(\xi) = s(\eta)\}.$$

Let Δ denote the diagonal in $X \times X$. Its closure $\overline{\Delta} \subseteq \beta(X\times X)$ is a clopen set, and so by the assumed continuity of m, $m^{-1}(\overline{\Delta}) = \overline{m^{-1}(\Delta)}$ is a clopen subset of K. It now follows from Proposition 6.3 that, if $L = (X\times X) \times_X (X \times X) = \{((x,y),(y,z)) : x, y, z \in X\}$, then $M = m^{-1}(\Delta) \cap L$ is decomposable relative to L.

Now M is the set $\{((x,y),(y,x)) : x, y \in X\}$, and decomposability implies that there are finitely many subsets A_i and B_i of $X\times X$ such that $M = \bigsqcup_{i=1}^{n} (A_i \times B_i) \cap L$. Let us then fix a point $y \in X$; for each $x \in X$ the point $((x,y),(y,x))$ belongs to M, and therefore there is an index i_x such that $(x,y) \in A_{i_x}$ and $(y,x) \in B_{i_x}$. If $i_x = i_{x'} = i$ for some x, x' then $((x,y),(y,x')) \in (A_i \times B_i) \cap L$ and thus $x = x'$; we conclude that the map $x \mapsto i_x$ is injective from X to $\{1, \dots, n\}$. Hence X is finite $\blacksquare$

6.3 The translation groupoid of a coarse space

Let X be a set. A *partial bijection* of X is a triple (D, R, t) consisting of two subsets D and R of X together with a bijection $t\colon D \to R$. Usually we denote such a partial bijection by the single letter t and refer to D as the *domain* of t and R as the *range*. Two partial bijections t and t' are *composable* if the range of t' equals the domain of t, in which case the composite tt' is defined in the obvious fashion.

The *graph* of a partial bijection t is the subset of $X \times X$ consisting of all points $(x, t(x))$, with x in the domain of t.

(6.6) LEMMA: *Let* X *be a set and let* $E \subseteq X \times X$ *be the union of finitely many graphs of partial bijection. Then each subset of* E *is realtively decomposable. Consequently, the continuous extension of* $(s, t)\colon E \to X \times X$ *injects* $\overline{E}$ *into* $\beta X \times \beta X$.

PROOF: By Proposition 6.4 we must show that each subset of E is relatively decomposable.

Let $A \subseteq E$. If the restriction of s to A is injective, then $A = (s(A) \times t(A)) \cap E$ is relatively decomposable. This remark already addresses the case when E consists of the graph of a single partial translation.

Suppose that E is the union of the graphs of partial bijections t_i, $i = 1, \ldots, n$, with domains D_i and ranges R_i. Define a graph[17] whose vertices are the points of X, and such that x', x'' are linked by an edge if and only if there is an $x \in X$ and indices i', i'' for which $x' = t_{i'}(x)$, $x'' = t_{i''}(x)$.

This graph has vertex degree at most $n-1$ by construction. By a greedy algorithm, it has a vertex-coloring with n colors; that is to say, one can write $X = \bigsqcup_{j=1}^{n} X_j$ in such a way that the restriction of the map s to $(X \times X_j) \cap E$ is injective. Thus any subset A of E can be written as the union of finitely many subsets $A_j = (X \times X_j) \cap A$, to each of which the restriction of s is injective; our earlier remark now completes the proof. ■

(6.7) LEMMA: *A subset* $E \subseteq X \times X$ *is the union of the graphs of finitely many partial bijections if and only if it is* uniformly proper*, which is to say that the cardinalities of the sections* $E_x = \{y : (x, y) \in E\}$ *and* $E^y = \{x : (x, y) \in E\}$ *are uniformly bounded. The union may always be taken to be disjoint.*

PROOF: Let E be uniformly proper, with the cardinalities of the sections bounded by n. Arguing as in the proof of the previous lemma divide E into a disjoint union of $E_1, \ldots, E_n$ such that the restriction of s to each E_i is injective; and also into a disjoint union of $E^1, \ldots, E^n$ such that the restriction of t to each E^j is injective. Then the $E_i \cap E^j$ are the graphs of n^2 partial bijections and their union is E. ■

For the remainder of this section let X be a coarse space. We will assume that X is of *discrete bounded geometry*, by which we mean that each controlled set is uniformly proper. For example, the metric coarse structure on a bounded geometry, uniformly discrete metric space.

(6.8) REMARK: Removing the discreteness hypothesis — in a functorial way — appears to be a nontrivial matter.

(6.9) DEFINITION: *Let* X *be a coarse space of discrete bounded geometry. The* translation groupoid *of* X *is the subset* $G(X) \subseteq \beta(X \times X)$ *which is the union of the closures of all the controlled subsets.*

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[17] In the graph-theory sense, i.e. a 1-dimensional simplicial complex.

In calling $G(X)$ a groupoid we have anticipated the main result of this section:

(6.10) THEOREM: *Let X be a coarse space of discrete bounded geometry. The source, target, inverse and multiplication maps on the pair groupoid $X \times X$ have unique continuous extensions to $G(X)$. With respect to these extensions, $G(X)$ becomes a principal, étale, locally compact Hausdorff topological groupoid with object set βX.*

Here are some preliminary remarks. The closures of the controlled sets are clopen in $\beta(X \times X)$, so their union $G(X)$ is open. As an open subset of a compact Hausdorff space, it is locally compact Hausdorff. The maps $s, t \colon \beta(X \times X) \to \beta X \times \beta X$ are extended by continuity, and by the preceding lemmas the pair $(s, t) \colon \overline{E} \to \beta X \times \beta X$ is injective on the closure of any controlled set E. Since the controlled sets (and therefore their closures!) form a directed system under inclusion, it follws that (s, t) is injective on $G(X)$ which is therefore principal. The inverse map $(x, y) \mapsto (y, x)$ is certainly continuous (even on $\beta(X \times X)$). It remains to prove therefore that multiplication can be extended continuously (this is the main point), and that s, t are local homeomorphisms. Note that since $X \times X$ is dense in $G(X)$, the continuous extensions of the operations are certainly unique if they exist.

(6.11) DEFINITION: *A* partial translation *of a coarse space X is a partial bijection whose graph is controlled.*

(6.12) DEFINITION: *Let X be a coarse space, and let ω be an ultrafilter of subsets of $X \times X$. Call ω an* ultratranslation *if it is generated by the graphs of partial translations; in other words, given any $S \in \omega$, there is $G \in \omega$ with $G \subseteq S$ and G the graph of a partial translation.*

In other words, ω is an ultratranslation if it has a *filter base* comprising the graphs of partial translations.

(6.13) PROPOSITION: *Let X be a coarse space of discrete bounded geometry. Then the points of $G(X)$ are identified with the ultratranslations of X.*

PROOF: The points of $\beta(X \times X)$ are identified with ultrafilters of subsets of $X \times X$, so what we have to show is that such a point belongs to the closure of a controlled set E if and only if the corresponding ultrafilter is an ultratranslation.

The point $x \in \beta(X \times X)$ is the intersection of the closures of all those S belonging to the corresponding ultrafilter ω. Thus if ω contains the graph of even one partial translation, then x belongs to $G(X)$.

Suppose that x belongs to the closure of a controlled set E and let ω be the corresponding ultrafilter of subsets of E. By Lemma 6.7, E is the disjoint union of the graphs of finitely many partial translations. One of these graphs and one only, say G, belongs to ω. Now if $S \in \omega$, then $S \cap G \in \omega$ also and it is the graph of a partial translation. Thus ω is an ultratranslation. ■

(6.14) REMARK: We can easily describe the maps $s, t\colon G(X) \to \beta X$ in this language. In fact, let ω be an ultratranslation. The collection of all the domains of the partial translations representing ω is a base for an ultrafilter of subsets of X; this ultrafilter represents the point $s(\omega) \in \beta X$. Similarly for the map t.

Let ω' and ω'' be ultratranslations of X, with $s(\omega') = t(\omega'')$. Let Ω be the collection of all partial translations t of X obtained as the composites $t't''$, where t', t'' are partial translations belonging to ω', ω'' respectively and the range of t'' is equal to the domain of t'. There are many such composable pairs.

In slightly more detail, let t' and t'' be *any* partial translations representing ω' and ω''. Let D' be the domain of t', R'' the range of t''. Both these sets belong to the ultrafilter of subsets of X that represents $s(\omega') = t(\omega'')$; therefore, so does their intersection. Let u' be the restriction of t' to $D' \cap R''$, and let u'' be the restriction of t'' to $(t'')^{-1}(D'' \cap R')$. Then u' represents ω', u'' represents ω'', and they are composable. Let us call their composite the *maximal composite* of t' and t''.

(6.15) LEMMA: *With notation as above, the set Ω of composites of representatives of ω' and ω'' is a basis for an ultratranslation ω of X.*

PROOF: We may assume without loss of generality that the graphs of all t' belong to a controlled set E', and the graphs of all t'' belong to a controlled set E''. Thus the graphs of all the $t \in \Omega$ belong to the controlled set $E = E' \circ E''$.

It is easy to see that the intersection of any two members of Ω includes another member of Ω; thus Ω is a base for a filter. To show that Ω is a base for an *ultrafilter*, we must show that for each subset S of E, either S or $E \setminus S$ includes a member of Ω. Because each subset of E is relatively decomposable (Lemmas 6.6 and 6.7) it suffices to establish this dichotomy when $S = (A \times B) \cap E$, with $A, B \subseteq X$.

Now, either A or $X \setminus A$ belongs to the ultrafilter (of subsets of X) that represents $s(\omega'')$, and either B or $X \setminus B$ belongs to the ultrafilter (of subsets of X) that represents $t(\omega')$. The proof thus divides into four cases. Consider for instance the case when A represents $s(\omega'')$ and B represents $t(\omega')$. Then A has a subset which is the domain of a t'' representing ω'', and B has a subset which is the range of a t' representing ω'. The maximal composite (see the small-print remark above) of t' and t'' is defined, it belongs to Ω, and its graph is plainly a subset of $A \times B$.

In the other three possible cases we can argue similarly to find a partial translation in Ω whose graph is contained in the complement of $A \times B$.

Finally, the resulting ultrafilter is an ultratranslation because it is made up of subsets of the controlled set E. ■

(6.16) DEFINITION: *We define the* product *of two ultratranslations* ω' *and* ω'' *with* $s(\omega') = t(\omega'')$ *to be the ultratranslation generated by the compositions of representatives for* ω' *and* ω''*, as in the preceding lemma.*

This product defines an operation $m\colon G(X) \times_{\beta X} G(X) \to G(X)$, and brief reflection shows that m extends the multiplication of the pair groupoid $X \times X$.

(6.17) LEMMA: *The product operation on ultratranslations is continuous.*