Math 529 Homework 4: Due November 11

Question 1. Let K be an Eilenberg-MacLane space of type $K(\mathbb{Z}, n)$. Show that $H^n(K; \mathbb{Z}) = \mathbb{Z}$. Fixing a generator, obtain a homomorphism

$$[X, K] \to H^n(X; \mathbb{Z})$$

for any finite complex X. Show that this homomorphism is actually an *isomorphism*. (Actually, I'll give full credit if you just show it's surjective, but isomorphism is the correct theorem.)

Question 2. Show that if m is odd then $\mathbb{C}P^m$ is diffeomorphic to the total space of a circle bundle over a quaternionic projective space. Deduce that $\mathbb{C}P^m$ is cobordant to zero.

Question 3. Let M be a compact oriented manifold (of dimension n) and let $\Delta \in M \times M$ be the diagonally embedded copy of M. Let $u \in H^n(M \times M; \mathbb{R})$ be the cohomology class Poincaré dual to Δ. Show that

$$u = \sum (-1)^{\dim b_i} b_i \times b_i^*$$

where b_i is a basis for $H^*(M; \mathbb{R})$, b_i^* is the dual basis with respect to the intersection form, and $\times$ denotes the external product $H^*(M) \times H^*(M) \to H^*(M \times M)$.

Question 4. If V is a k-dimensional oriented real vector bundle over X, the *Euler class* $e(V) \in H^k(X)$ is the restriction of the Thom class (in $H^k_c(V)$) to X (considered as the zero section of V).

Using the previous question, show that if M is a compact oriented manifold, then the Euler characteristic of M (the alternating sum of the Betti numbers) is equal to $\langle e(TM), [M] \rangle$. Hint: Show that the normal bundle to Δ in $M \times M$ is isomorphic to the tangent bundle of M.