Math 529 Homework 3: Due October 17

Question 1. Let X be a connected CW-complex with fundamental group $\pi_1(X) = \Gamma$ and with $\pi_i(X) = 0$ for $1 < i < n$. Show that

$$H_n(X)/h_*(\pi_n(X)) = H_n(K)$$

where h is the Hurewicz homomorphism and K is an Eilenberg-MacLane space of type $(\Gamma, 1)$.

Question 2. Let p and q be distinct primes. Show that for any $m, n \geq 2$ the smash-product of Eilenberg-MacLane spaces

$$K(\mathbb{Z}_p, m) \wedge K(\mathbb{Z}_q, n)$$

is contractible (or weakly equivalent to a point, if our spaces are not CW-complexes). Hint: Use the Künneth Theorem.

Question 3. Let $X = S^1$ and let L be the unique non-trivial real line bundle over X (the 'Möbius band bundle'). Let Γ be the sheaf of sections of L (i.e. $\Gamma(U)$ is the space of sections of L restricted to U). Calculate $H^*(X; \Gamma)$.

Question 4. The *Alexander-Spanier* cohomology of a space X is defined as follows. Consider the collection of all functions $X^{q+1} \to G$, where G is some fixed abelian group. Such a function f is called *locally zero* if there is an open cover $\mathcal{U}$ of X such that $f(x_0, \ldots, x_q) = 0$ if $x_0, \ldots, x_q$ all belong to one of the members of $\mathcal{U}$. The quotient group of all functions $X^{q+1} \to G$ divided by the subgroup of locally zero functions is the Alexander-Spanier cochain group $C^q(X; G)$. These groups are equipped with a coboundary

$$\delta \colon C^q \to C^{q+1} : (\delta f)(x_0, \ldots, x_{q+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, \widehat{x_i}, \ldots, x_{q+1})$$

and the cohomology of the resulting complex is the Alexander-Spanier cohomology of X.

Show (I finally got round to the question) that the Alexander-Spanier cohomology is always isomorphic to the Cech cohomology.