Math 529 Homework 2: Due October 3

Question 1. Let $X = \mathbb{CP}^\infty$ denote infinite dimensional complex projective space (that is, the direct limit of

$$\mathbb{CP}^1 \to \mathbb{CP}^2 \to \cdots$$

equipped with the weak topology). Compute the homology of X

- by equipping it with an explicit CW complex structure, and
- by considering the Serre spectral sequence of a suitable fibration $S^1 \to S^\infty \to X$.

If you wish, repeat the procedure for quaternionic projective space.

Question 2. Let $F \to E \to B$ be a fibration. Show that if we make the inclusion $F \to E$ into a fibration by Serre's construction from the previous question sheet, the fiber of *this* fibration is homotopy equivalent to ΩB.

Question 3. Let $F \to E \to B$ be a fibration, with B simply connected and F connected. Consider the Serre spectral sequence. There are surjections $H_q(F) = E^2_{0q} \to E^3_{0q} \to \cdots \to E^\infty_{0q}$ and an injection $E^\infty_{0q} \to H_q(E)$. Show that the composite of these maps (called the *edge homomorphism* for the spectral sequence) is the map induced on homology by the inclusion $F \to E$. (Use the naturality of the spectral sequence.)

State and prove an analogous result for the groups $E^\bullet_{p0}$.

Question 4. Consider a map $S^k \to S^k$ of degree n (assume that $k > 2$). Make this map into a fibration by Serre's construction. Compute the homology of the fiber.

Question 5. A *double complex* is a list of abelian groups C_{pq}, $, p, q \geq 0$, equipped with two differentials $d\colon C_{pq} \to C_{p-1,q}$ and $\delta\colon C_{pq} \to C_{p,q-1}$, satisfying $d^2 = 0$, $\delta^2 = 0$, and $d\delta + \delta d = 0$. Suppose that $H_{pq}(C, d) = 0$ for $p > 0$ and $H_{pq}(C, \delta) = 0$ for $q > 0$. Show that

$$H_{0n}(C, d) \cong H_{n0}(C, \delta).$$

(Consider two spectral sequences associated to different filtrations of the *total complex* with groups $T_n = \bigoplus_{p+q=n} C_{pq}$ and differential $d + \delta$.)