Math 529 Homework 1: Due September 16

Question 1. Let X, Y and Z be topological spaces, with Y and Z locally compact Hausdorff. Prove the *exponential law for function spaces*: the natural map

$$(X^Y)^Z \to X^{Y\times Z}$$

is a homeomorphism. (All function spaces have the compact-open topology.)

Question 2. The *suspension* of a pointed space X is the quotient space

$$SX := S^1 \times X/(S^1 \times \{*\} \cup \{*\} \times X)$$

where we use the letter $*$ to denote the base point in the two spaces. Show that there is a natural identification of homotopy sets

$$[SX, Y] = [X, \Omega Y]$$

for any (locally compact) X and Y.

Question 3. Let $f\colon X \to Y$ be a map. Let E be the subset of $X \times Y^I$ consisting of pairs (x, γ) such that $f(x) = \gamma(0)$. Show that there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & E \\ & \searrow^{f} & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where i is a homotopy equivalence and p is a fibration. Thus "up to homotopy, every map is a fibration". (This is a theorem of Serre.)

Question 4. Let X be the one-point union of S^1 and S^2 (usually denoted $S^1 \vee S^2$). Show that the abelian group $\pi_2(X)$ is not finitely generated. (Hint: Relate $\pi_2(X)$ to $\pi_2(\tilde{X})$, where $\tilde{X}$ denotes the universal cover of X.)

Question 5. Let $p\colon E \to B$ be a fibration. A *section* for p is a map $s\colon B \to E$ such that $ps = 1_B$. Show that if a fibration has a section then

$$\pi_k(E) \cong \pi_k(B) \times \pi_k(F), \quad k \geq 2$$

where F is the fiber.

Extra credit Show that the closed 4-manifolds $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ have the same homotopy groups in every dimension, but are not homotopy equivalent. (This is HARD.)