Math 502 Homework 9 Due Friday, April 4th

(1) Let K be a closed subset of the Riemann sphere $\mathbb{S}$, whose complement is connected and nonempty. Let $p \in \mathbb{S} \setminus K$. Show that any function f holomorphic on a neighborhood of K can be approximated, uniformly on K, by rational functions with poles only at p. (Hint: Use a suitable conformal transformation to reduce this to the version of Runge's theorem that we proved in class.)

(2) Let Ω be the set $\{x + iy \in \mathbb{C} : x > -1, |y| < e^{-e^x}\}$. Using the previous question, show that there is a sequence of rational functions f_n on $\mathbb{C}$ having the following properties:

- $f_1(z) = 10/(z-2)$,
- for $n > 1$, $f_n(z)$ is a rational function with pole only at $n+1$, and
- $|f_{n+1}(z) - f_n(z)| < 2^{-n}$ for all $z \notin \Omega$ and for all z with $\mathrm{Re}(z) \leqslant n$.

Deduce that the functions f_n converge uniformly on compact subsets of $\mathbb{C}$ to a nonconstant entire function which is bounded on the complement of Ω.

(The function grows extremely rapidly along Ω itself.)

(3) Let $E = \ell^\infty$ be the space of all *bounded* sequences $\mathbf{a} = \{a_n\}$ of complex numbers. Show that there is a continuous linear functional $\phi\colon E \to \mathbb{C}$ which has the property that if $\lim a_n$ exists, then $\phi(\mathbf{a}) = \lim a_n$.

Let $B\colon E \to E$ take the sequence $\{a_n\}$ to the sequence $\{b_n\}$ where

$$b_n = \frac{a_1 + \cdots + a_n}{n}.$$

Show that $B\colon E \to E$ is a bounded linear operator. Show that $\psi = \phi \circ B$ is a linear functional on E which has the property that if $\lim a_n$ exists, then $\psi(\mathbf{a}) = \lim a_n$. In addition, show that $\psi(\mathbf{a}) = \psi(\mathbf{a}')$, where $a'_n = a_{n+1}$. (ψ is called a *Banach limit*.)