Math 502 Homework 5 Due Friday, February 22nd

(1) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2 \cos x}{4 + x^4} dx$$

by means of contour integration. Be sure to justify carefully all the steps in your argument.

(2) By contour integration find the value of

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 5x + 6}\, dx.$$

Be sure to justify carefully all the steps in your argument.

(3) A theorem of Carlson states that if $f(z)$ is holomorphic and bounded in $\Omega = \{z : \mathrm{Im}(z) > -\epsilon\}$, $\epsilon > 0$, and if moreover $f(z) = 0$ whenever $z = ki$, $k = 1, 2, \ldots$, then f is identically zero. Complete the following outline to prove this theorem:

(a) Show that for any $a \in \mathbb{R}^+$ and any $n = 1, 2, \ldots$ we have

$$f(ia) = \frac{i^n (a-1) \cdots (a-n)}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{(x - ia)(x - i)(x - 2i) \cdots (x - ni)} dx.$$

(b) Show that for $a > 1$ the absolute value of the integral appearing above is no more than $M\pi/n!$, where M is an upper bound for $|f|$ in Ω.

(c) By letting $n \to \infty$ deduce from (a) above that $f(ia) = 0$ for all $a > 1$.

Carlson's theorem now follows by analytic continuation.