Math 502 Homework 12 Due Friday, April 26th

(1) Let T be a compact, self-adjoint operator on some Hilbert space. Suppose that there exists a polynomial p such that $p(T) = 0$. Prove that the range of T is finite-dimensional.

(2) Let T be a compact operator on an (infinite-dimensional) Hilbert space and suppose that $\ker(T) = 0$. Show that there is a sequence of $\{S_n\}$ of bounded operators such that, for all $x \in H$, $S_n T x \to x$ as $n \to \infty$. (Hint: consider first the case T self-adjoint, and use the spectral theorem.)

Is it possible to choose the operators S_n so that $S_n T$ tends to the identity in the operator norm?

(3) Consider the linear mapping T from $L^2[0, \infty)$ to itself defined by

$$Tf(x) = \begin{cases} e^{-x} f(x-1) & (x \geqslant 1) \\ 0 & (x < 1) \end{cases}$$

Show that the spectrum of T is the set $\{0\}$.