

Math 502 Homework 1 Due Friday, January 25th

(1) Let $f = u + iv$ be a holomorphic function of $z = x + iy$. Show that the function $g = \log |f|^2 = \log(u^2 + v^2)$ satisfies Laplace's equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$$

in a neighborhood of any point where f is nonzero. (You may assume that u and v are twice continuously differentiable.)

(Try to do this in two different ways: once by a brute force calculation using the Cauchy-Riemann equations and once by using the properties of the locally defined holomorphic logarithm function, discussed in lecture 5)

(2) Let a be a complex number with $|a| < 1$. Prove that the mapping

$$z \mapsto \frac{z - a}{1 - \bar{a}z}$$

is a holomorphic bijection (with holomorphic inverse) from the unit disc $U = \{z : |z| < 1\}$ onto itself.

(3) Let z be a complex number with positive real part. By induction on n , show that

$$\int_0^1 t^{z-1}(1-t)^n dt = \frac{n!}{z(z+1)\cdots(z+n)}.$$

Substitute $t = u/n$ and let $n \rightarrow \infty$ to obtain

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^{z-1}}{z(z+1)\cdots(z+n-1)},$$

where Γ is the Gamma function defined in class. (Do your best to justify the limit processes that are used in this argument.)