Lecture 1
The complex plane

Exercise 1.1. Show that the modulus obeys the *triangle inequality*

$$|z \pm w| \leqslant |z| + |w|.$$

This allows us to make the complex plane into a metric space, and thus to introduce topological notions such as open and closed sets, continuity. etc.

Solution. Writing $z = a + bi$, $w = c + di$ we need to prove

$$(a+c)^2 + (b+d)^2 \leqslant \left(\sqrt{a^2+b^2} + \sqrt{c^2+d^2}\right)^2.$$

Square both sides and cancel terms; it's enough to show

$$ac + bd \leqslant \sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

Square and cancel again; enough to show

$$2abcd \leqslant a^2d^2 + b^2c^2.$$

But right minus left is $(ad - bc)^2 \geqslant 0$, so this is true. ☐

Exercise 1.2. Show that the dot product of two complex numbers z and w (considered as vectors in $\mathbb{R}^2$) is the real part of $z\bar{w}$. Use this to give another proof that multiplication by a fixed complex number is a conformal linear transformation.

Solution. If $z = x + yi$ and $w = p + qi$ then

$$z\bar{w} = (x+yi)(p-qi) = (xp+yq) + (yp-xq)i.$$

The real part of this is the dot product of z and w, thought of as vectors in $\mathbb{R}^2$. Now let w and w' be complex numbers. By what we just remarked, the angle between them is

$$\cos^{-1}\left(\frac{\operatorname{Re} w'\bar{w}}{|w||w'|}\right).$$

The angle between zw and zw' is

$$\cos^{-1}\left(\frac{\operatorname{Re} w'z\bar{z}\bar{w}}{|zw||zw'|}\right).$$

Since $z\bar{z} = |z|^2$ the factors of z cancel and the angles are equal. ☐

Lecture 2

Elementary Topology and Analysis in the Complex Plane

Exercise 2.1. If f is differentiable at every point of a connected open subset Ω of $\mathbb{C}$, and $f'(z) = 0$ for all $z \in \Omega$, show that f is constant on Ω.

Solution. Let $\gamma\colon [0,1] \to \Omega$ be a differentiable path. Then $f \circ \gamma\colon [0,1] \to \mathbb{C}$ is differentiable and has derivative zero, hence it is constant by the usual form of the Mean Value Theorem (applied separately to the real and imaginary parts). It follows that f is constant along every path in Ω. Since Ω is connected, it is path connected, so f is constant on Ω. $\square$

Lecture 3

Holomorphic functions and power series

Exercise 3.1. Prove Hadamard's formula for the radius of convergence, which is

$$1/r = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Solution. Make use of the following property of the limit superior: if $b = \limsup b_n$, then for any $\epsilon > 0$, only finitely many b_n are greater than $b + \epsilon$, and infinitely many b_n are greater than $b - \epsilon$.

Applied to Hadamard's formula, this says that for any $R > r$ there are infinitely many n for which $|a_n| > 1/R^n$; whereas for any $\rho < r$, the inequality $|a_n| < 1/\rho^n$ is satisfied for all but finitely many n.

Thus if $|z| > R$ the series $\sum a_n z^n$ has infinitely many terms of absolute value greater than 1, so cannot converge. It follows that the radius of convergence is less than or equal to R. If $|z| < \rho$ then all but finitely many terms of $\sum a_n z^n$ are dominated by the corresponding terms of the convergent geometric series $\sum |z|^n/\rho^n$. It follows that the radius of convergence is greater than or equal to ρ. Since ρ, R are arbitrary subject to the condition $\rho < r < R$, we conclude that the radius of convergence is r. ☐

Lecture 4

Some examples of holomorphic functions

Exercise 4.1. Show that the zeta function can also be defined for $\operatorname{Re} s > 1$ by the integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt.$$

Solution. For $t > 0$ we can write

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = \sum_{n=1}^{\infty} e^{-nt}.$$

From the dominated convergence theorem, then,

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = \sum_{n=1}^{\infty} \int_0^\infty t^{s-1} e^{-nt} dt.$$

But the substitution $u = nt$ shows that

$$\int_0^\infty t^{s-1} e^{-nt} dt = n^{-s} \int_0^\infty u^{s-1} e^{-u} du = n^{-s} \Gamma(s)$$

and so

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1} dt = \sum_{n=1}^{\infty} n^{-s} \Gamma(s) = \Gamma(s)\zeta(s),$$

as required. □

Exercise 4.2. Fill in the details of this proof.

Lecture 5

Plane topology and complex analysis

Exercise 5.1. Prove that there is no branch of the logarithm function defined on $\mathbb{C}\setminus\{0\}$.

Solution. If a holomorphic logarithm function existed on the whole of $\mathbb{C}\setminus\{0\}$, it would in particular give a continuous map from $\mathbb{T}$ to $i\mathbb{R}$ (the imaginary axis) whose composite with the exponential map $i\mathbb{R} \to \mathbb{T}$ would be the identity. But since $i\mathbb{R}$ is contractible, the induced map on fundamental groups

$$\pi_1(\mathbb{T}) \to \pi_1(i\mathbb{R}) = 0 \to \pi_1(\mathbb{T})$$

would be the zero map, and this is impossible.

Alternatively, argue that a holomorphic logarithm function g would give rise to a holomorphic square root function $e^{g/2}$ and invoke the previous result. □

Lecture 6

Path integrals and Cauchy's theorem

Exercise 6.1. Show that Cauchy's theorem for a triangle remains true if f is continuous in Ω and differentiable everywhere except at a single point. (Approximate a triangle with the 'bad' point as vertex by nearby triangles that don't contain the bad point.)

Solution. First of all, consider the case where the 'bad' point is the vertex A of a triangle ABC. Since f is continuous at A, given $\epsilon > 0$ we can find points B' (on AB) and C' (on AC) sufficiently close to A that

$$\left| \int_{\partial AB'C'} f(z)dz \right| < \epsilon.$$

Now

$$\int_{\partial ABC} f(z)dz = \int_{\partial AB'C'} f(z)dz + \int_{\partial BB'C} f(z)dz + \int_{\partial CB'C'} f(z)dz$$

and the last two integrals vanish by Cauchy's theorem (original version). Thus

$$\left| \int_{\partial ABC} f(z)dz \right| < \epsilon$$

and letting $\epsilon \to 0$ we get the result.

If the bad point is on an edge or inside the triangle, subdivide into two or three smaller triangles with the bad point at the vertex and apply the previous result. □

Lecture 7

Consequences of Cauchy's theorem

Exercise 7.1. (The reflection principle) Let f be continuous on the closed upper half-plane, holomorphic on the open upper half-plane, and real-valued on the real axis. Show that, if we extend f to the whole of $\mathbb{C}$ by defining

$$f(z) = \overline{f(\bar{z})}$$

when z is in the lower half-plane, then the extended function is holomorphic on the entire complex plane.

Solution. We shall use Morera's theorem. If T is a triangle in the open upper half-plane, then $\int_{\partial T} f(z)dz = 0$ by cauchy's theorem. This result extends to triangles in the closed upper half-plane, since f is continuous and any triangle in the closed upper half-plane can be regarded as the limit of a sequence of triangles in the open upper half-plane.

If T is a triangle in the open lower half-plane then

$$\int_{\partial T} f(z)dz = \int_{\partial T} \overline{f(\bar{z})}dz = \overline{\int_{\partial \overline{T}} f(w)dw} = 0$$

using the substitution $w = \bar{z}$. Again, this extends by continuity to triangles in the closed lower half-plane.

Any triangle can be subdivided into a sum of triangles in the upper and lower half planes. Thus the integral of f around any triangle is zero. Morera's theorem now implies that f is holomorphic. $\square$

Lecture 8

The rigidity of holomorphic functions

Exercise 8.1. Show that if f is an entire function and for each $z \in \mathbb{C}$ there is n such that $f^{(n)}(z) = 0$, then f is in fact a polynomial.

Solution. Suppose that f is *not* a polynomial. Then each derivative $f^{(n)}$ is a non-constant entire function, and hence its zeroes form a discrete subset of $\mathbb{C}$. The set $U_n = \{z : f^{(n)}(z) \neq 0$ is therefore open and dense. By the Baire category theorem the intersection $V = \bigcap U_n$ is dense as well, which contradicts the hypothesis (each point $z \in V$ has $f^{(n)}(z) \neq 0$ for all n). $\square$

Lecture 9

The Global Cauchy Theorem

Exercise 9.1. Show that every 1-cycle is equivalent to a formal linear combination of closed paths.

Solution. Let Γ be a 1-cycle, and let it be represented as $\sum_j k_j[\gamma_j]$. We can replace it by an equivalent cycle where all the numbers k_j are positive; to do this, notice that if $\gamma\colon [a,b] \to \mathbb{C}$ is a path, then the cycle $[\gamma]$ is equivalent to $-[\gamma^\natural]$, where $\gamma^\natural$ is defined by

$$\gamma^\natural(t) = \gamma(a+b-t).$$

Assume then that all the k_j are positive, and define the 'height' $|\Gamma|$ to be the sum of the k_j. We'll prove by induction of $|\Gamma|$ that every 1-cycle is equivalent to a combination of closed paths. This is apparent if $|\Gamma| = 1$ since then $\Gamma = [\gamma]$ and the cycle condition implies that the beginning and ending points of γ are the same.

Now given a general cycle $\Gamma = \sum k_j[\gamma_j]$ (with all k's positive), the cycle condition implies that each end point of one γ_j is the start point of another $\gamma_{j'}$. Thus, starting at γ_1, we can find a chain of γ's each of which begins where the previous one ends. Because there are only finitely many γ's this chain must close at some point, so there is a finite sequence of γ's — by renumbering we may assume that it is $\gamma_1, \ldots, \gamma_p$ — such that the end of γ_j is the start of γ_{j+1} for $1 \leqslant j \leqslant p-1$, and the end of γ_p is the start of γ_1.

But then $\Gamma' = [\gamma_1] + \cdots + [\gamma_p]$ is a cycle, and it is equivalent to the closed path obtained by concatenating $\gamma_1, \ldots, \gamma_p$. Moreover, $\Gamma - \Gamma'$ is a cycle, and its height $|\Gamma - \Gamma'| = |\Gamma| - p < \gamma$. By induction, $\Gamma - \Gamma'$ is equivalent to a sum of closed paths; so $\Gamma = (\Gamma - \Gamma') + \Gamma'$ is equivalent to a sum of closed paths as well. $\square$

Exercise 9.2. Give an example of a cycle (in fact a closed path) in $\Omega = \mathbb{C} \setminus \{0,1\}$ that is homologous to zero but is not *homotopic* to zero (i.e., cannot be continuously deformed to a constant path).

Lecture 10

Laurent series and the residue theorem

Exercise 10.1. Derive the formula for the residue at a double pole: if g has a double zero at a, then the residue of f/g at a is

$$2\left(\frac{f'(a)}{g''(a)} - \frac{f(a)g'''(a)}{3g''(a)^2}\right).$$

Verify that this is consistent with our solution to Example **??**.

Solution. Write

$$g(z) = (z-a)^2 h(z)$$

where h is holomorphic and $h(a) \neq 0$. Then (either by direct calculation using the product rule for differentiation, or more simply by comparing Taylor series on the left and right hand sides of the above identity) we find

$$h(a) = \tfrac{1}{2}g''(a), \quad h'(a) = \tfrac{1}{6}g'''(a).$$

Now write

$$\frac{f(z)}{g(z)} = \frac{f(z)/h(z)}{(z-a)^2}.$$

The numerator is holomorphic near a, so the residue is

$$\left[\frac{d}{dz}\frac{f(z)}{h(z)}\right]_{z=a} = \frac{f'(a)}{h(a)} - \frac{f(a)h'(a)}{h(a)^2} = \frac{2f'(a)}{g''(a)} - \frac{2f(a)g'''(a)}{3g''(a)^2}$$

as required. ☐

Lecture 11

Poles and Zeroes

Exercise 11.1. Suppose that f_1 and f_2 are as above, possibly having zeroes or poles at a. What, if anything, can you say about the order of the zero or pole of $f_1 + f_2$ at a, in terms of the orders of f_1 and f_2?

Solution. Let f_1 and f_2 have poles of orders N_1 and N_2 respectively. If $N_1 \neq N_2$ then ‘the worst singularity wins’: $f_1 + f_2$ has a pole of order $\max\{N_1, N_2\}$. However, if $N_1 = N_2$, then the sum $f_1 + f_2$ may have a pole of order N_1 or of any lesser order (including no pole at all). These statements are easily verified by adding Laurent series. $\square$

Lecture 12

Calculations with the residue theorem

Exercise 12.1. Prove the (Fourier integral) formula

$$\int_{ai-\infty}^{ai+\infty} e^{-\pi u z^2 + 2\pi i w z}\, dz = u^{-1/2} e^{-\pi w^2/u}$$

for u real and positive, and $w \in \mathbb{C}$.

Solution. Complete the square to write the given integral as

$$\int_{ai-\infty}^{ai+\infty} e^{-\pi u (z - iw/u)^2} \cdot e^{-\pi w^2/u}\, dz = u^{-1/2} e^{-\pi w^2/u} \int_{bi-\infty}^{bi+\infty} e^{-\pi s^2}\, ds$$

where $s = u^{1/2}(z - iw/u)$ and $b = a - \operatorname{Re} w/u^{1/2}$. It suffices therefore to prove that

$$\int_{bi-\infty}^{bi+\infty} e^{-\pi s^2}\, ds = \int_{-\infty}^{+\infty} e^{-\pi s^2}\, ds$$

since the latter integral is 1 from our discussion of the Γ function. To see this, integrate the holomorphic function

$$s \mapsto e^{-\pi s^2}$$

around a rectangular contour with vertices at $\pm L$ and $\pm L + ib$. The integrals along the vertical sides tend to zero as $L \to \infty$, so the contour integral tends to

$$\int_{bi-\infty}^{bi+\infty} e^{-\pi s^2}\, ds - \int_{-\infty}^{+\infty} e^{-\pi s^2}\, ds.$$

But the contour integral is zero by Cauchy's theorem. $\square$

Lecture 13

The Riemann zeta function

Exercise 13.1. Using the functional equation and the Euler product formula, show that the only zeroes of $\zeta(s)$ outside the *critical strip* $0 \leqslant \operatorname{Re} s \leqslant 1$ are at the negative even integers. (These are the so-called *trivial zeroes* of ζ.)

Solution. First consider the case $\operatorname{Re} s > 1$. In this region we have the Euler product formula

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

The infinite product converges *absolutely* and the individual terms are all non-zero. Thus the product is also non-zero,

Now consider the case $\operatorname{Re} s < 0$. From the reflection formula

$$\zeta(s) = \frac{\pi^{s-1/2}\Gamma((1-s)/2)\zeta(1-s)}{\Gamma(s/2)}.$$

Recall also the infinite product for the Γ function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right].$$

This shows that Γ has no zeroes and has simple poles at 0, -1, -2,$\ldots$ Substituting this information into the reflection formula we see that the numerator of the fraction above has no zeroes in $\operatorname{Re} s < 0$ and the denominator has poles only at $s = -2, -4, -6, \ldots$. Thus these are exactly the zeroes of ζ in this range. $\square$

Lecture 14

Multi-valued functions

Exercise 14.1. Find a region on which a branch of the inverse cosine function is defined. (Start by proving that

$$\cos^{-1}(z) = i \log\left(z + \sqrt{z^2 - 1}\right)$$

and then look for a branch of the function on the right.)

Solution. If $z = \cos w = \frac{1}{2}(e^{iw} + e^{-iw})$, then e^{-iw} is a root of the quadratic equation $t^2 - 2zt + 1 = 0$. The roots of this equation are

$$t = z \pm \sqrt{z^2 - 1},$$

which gives the formula $w = i \log\left(z + \sqrt{z^2 - 1}\right)$ from the question. There are branch points of $\sqrt{z^2 - 1}$ at ± 1. Since $(z + \sqrt{z^2 - 1})(z - \sqrt{z^2 - 1}) = 1$, the expression under the logarithm sign is never zero, so no further branch points are introduced by the logarithm term. The most natural way to cut the plane here is to cut from $-\infty$ to -1 and 1 to ∞. The complement of these two cuts is a simply connected region, on which branches of $\sqrt{z^2 - 1}$ and then $i \log\left(z + \sqrt{z^2 - 1}\right)$ can be defined. Note that it does not suffice to cut from -1 to 1; the square root is well-defined on this cut region, but the logarithm is not. ☐

Lecture 15

More examples of contour integration

Lecture 16

The Riemann Sphere

Exercise 16.1. Let S be a compact connected Riemann surface. Show that the only holomorphic functions $S \to \mathbb{C}$ are the constant functions.

Solution. Let $f\colon S \to \mathbb{C}$ be holomorphic. Since S is compact, $|f|$ attains a maximum at some point $a \in S$. By the maximum principle (applied to $f \circ \phi^{-1}$, where ϕ is a chart near a), f must be constant (say equal to c) on some neighborhood of a. Consider now the set of all points $x \in S$ such that $f = c$ on a neighborhood of x. By definition this set is open. By the principle of isolated zeroes, it is closed. It is nonempty as we have seen, and thus (by connectedness) it is the whole of S. ☐

Lecture 17

Automorphisms

Exercise 17.1. The words "in general" in the argument above conceal an algebraic discussion which is necessary. What about the possibility that $p(z) - wq(z) = 0$ has a repeated root (repeated as many times as the degree)? Show that this cannot happen except for a finite set of values of w.

Solution. Suppose that for some nonzero value of w, the equation $p(z)-wq(z) = 0$ has a repeated root (for z). Then we have simultaneously $p'(z) - wq'(z) = 0$. Eliminating w, we get

$$p(z)q'(z) - q(z)p'(z) = 0.$$

The polynomial appearing on the left of this equation cannot be identically zero, because this would imply that p is a constant multiple of q. Therefore there are only finitely many values of z for which the displayed equation is satisfied, and each corresponds to at most one possible value of w. □

Exercise 17.2. Show that the automorphism groups of $\mathbb{S}$ and $\mathbb{C}$ are transitive.

Solution. Translations $z \mapsto z + c$ are automorphisms of $\mathbb{C}$, and it is obvious that the group of translations acts transitively on $\mathbb{C}$. Since translations also give automorphisms of $\mathbb{S}$, we see that any two finite points of $\mathbb{S}$ are equivalent under the automorphism group. The transformation $z \mapsto 1/z$ is also an automorphism of $\mathbb{S}$ and maps ∞ to a finite point, so in fact all points of $\mathbb{S}$ are equivalent. □

Lecture 18

Hyperbolic Geometry

Exercise 18.1. Prove the hyperbolic sine law: in a hyperbolic triangle as discussed above, one has

$$\frac{\sinh BC}{\sin\alpha} = \frac{\sinh CA}{\sin\beta} = \frac{\sinh AB}{\sin\gamma}.$$

Solution. Consider first the special case of a triangle with a right angle at C. Write $d(A,B) = c$, $d(B,C) = a$, $d(C,A) = b$.

We have

$$\begin{aligned} \cosh c &= \cosh a \cosh b \\ \cosh b &= \cosh a \cosh c - \sinh a \sinh c \cos\beta \end{aligned}$$

Rewrite the second equation to give

$$\sin^2\beta = 1 - \cos^2\beta = \frac{\sinh^2 a \sinh^2 c - (\cosh b - \cosh a \cosh c)^2}{\sinh^2 a \sinh^2 c}.$$

Substitute $\cosh a = \cosh c / \cosh b$ from the first equation and $\sinh^2 a = \cosh^2 a - 1$ (standard identity). After canceling a common factor of $\cosh^2 c - \cosh^2 b$ we get

$$\frac{\cosh^2 b - \cosh^2 c + \sinh^2 c}{\sinh^2 c} = \frac{\sinh^2 b}{\sinh^2 c}$$

so $\sin\beta = \sinh b / \sinh c$. This is the sine rule for a right triangle. The general case follows by dropping a perpendicular from a vertex of the triangle to the opposite side, and thus dividing it into two right triangles (exactly as in Euclidean geometry). $\square$

Lecture 19

The Arzela-Ascoli theorem

Exercise 19.1. Prove this.

Solution. Let $\mathcal{U}$ be the given cover. It has a Lebesgue number $\delta > 0$. By compactness, there is a finite cover of X by balls of radius $\delta/2$. Let $x_1, \dots, x_N$ be the centers of these balls. Let ψ_i, $i = 1, \dots, N$ be the function

$$\psi_i(x) = \begin{cases} \delta/2 - d(x, x_i) & (d(x, x_i) \leqslant \delta/2) \\ 0 & (d(x, x_i) \geqslant \delta/2) \end{cases}$$

For each i there is $U_i \in \mathcal{U}$ such that ψ_i is supported in U_i.[1] For each $U \in \mathcal{U}$ define

$$\phi_U = \left(\sum_{i : U_i = U} \psi_i \right) \Bigg/ \left(\sum_{i=1}^{n} \psi_i \right).$$

The denominator is > 0 everywhere because of the covering property, and clearly the ϕ_U form a partition of unity. $\square$

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[1] It is possible that $U_i = U_j$ even if $i \neq j$.

Lecture 20

The Riemann mapping theorem

Lecture 21

Constructing conformal maps

Lecture 22
Basics of Banach Spaces

Exercise 22.1. Show that the norm of linear maps is submultiplicative under composition, i.e. $\|S \circ T\| \leqslant \|S\|\|T\|$.

Solution. We have

$$\|STv\| \leqslant \|S\|\|Tv\| \leqslant \|S\|\|T\|\|v\|,$$

which gives the result. □

Lecture 23
The Hahn-Banach Theorem

Exercise 23.1. For any Banach space E construct an isometric injection $E \to E^{**}$.

Solution. Any $x \in E$ defines a linear map $e_x \colon E^* \to \mathbb{C}$ via the equation

$$e_x(\phi) = \phi(x).$$

Since $|e_x(\phi)| = |\phi(x)| \leqslant \|\phi\|\|x\|$, we see that e_x is a bounded linear functional on E^*, with norm $\|e_x\| \leqslant \|x\|$. Thus $e_x \in E^**$, and $x \mapsto e_x$ is a linear map from E to E^**.

Let us show that this linear map is an isometry (and so, in particular, that it is injective.) Let $x \in E$. By the Hahn-Banach theorem there is a linear functional $\phi \in E^*$, of norm 1, such that $|\phi(x)| = \|x\|$ (define ϕ first on the one dimensional subspace spanned by x, and extend by Hahn-Banach). Then $|e_x(\phi)| = \|x\|$ so $\|e_x\| \geqslant \|x\|$. We proved the opposite inequality above, so $\|e_x\| = \|x\|$ and $x \mapsto e_x$ is an isometric injection. □

Lecture 24

Applications of the Hahn-Banach Theorem

Lecture 25

Applications of Runge's Theorem

Exercise 25.1. Prove that the characteristic function of the set $A = \{z : (\operatorname{Re} z)(\operatorname{Im} z) = 0\}$ (that is, the union of the x and y axes) can be obtained as the pointwise limit of holomorphic functions on $\mathbb{C}$.

Solution. Let $A_n = A \cap \overline{D}(0; n)$ and let $B_n = \{z : |z| \leqslant n,\ |\operatorname{Re} z| \geqslant 1/n,\ |\operatorname{Im} z| \geqslant 1/n\}$. Apply Runge's theorem to $A_n \cup B_n$ to find a polynomial p_n that has $|p_n - 1| < 1/n$ on A_n, $|p_n| < 1/n$ on B_n. $\square$

Lecture 26

Convexity and the Hahn-Banach Theorem

Exercise 26.1. Give an example of an absorbing subset in a normed vector space which does *not* contain any neighborhood of the origin.

Solution. There are many possible solutions depending on how sophisticated you want to get. Here are a couple.

- Take $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$, and let A be the set

  $$\{re^{i\theta} : 0 < \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant \theta\}.$$

  This is easily seen to be absorbing, but it contains no disk around 0.

- Let E be the vector space of differentiable functions $[-1,1] \to \mathbb{R}$, with the supremum norm, and let $A = \{f : |f'(0)| < 1\}$. Then A is absorbing, but it contains no neighborhood of 0 (because one can have very small functions with very large derivatives).

Both of these examples feel a bit like cheating: in (a), the set A is not convex and in (b), the space E is not complete. Can one have a *convex* absorbing set in a *complete* space which is not a 0-neighborhood? The answer is yes. Indeed, if $\phi\colon E \to \mathbb{R}$ is a *discontinuous* linear functional, then $\{x \in E : |\phi(x)| < 1\}$ is an absorbing set that cannot contain a neighborhood of the origin. To produce such a linear functional on a complete space needs some transfinite machinery again. For instance, take $E = C[-1,1]$, F the subspace of functions differentiable at 0, and $\phi(f) = f'(0)$ for $f \in F$. Extend ϕ to E using the technique of the Hahn-Banach theorem (but without bothering about the norms at all). $\square$

Lecture 27

Weak topologies

Exercise 27.1. The argument above, as stated, works for *real* vector spaces only. Work out how to generalize it to the complex case.

Solution. If E is a complex vector space, and A and b are as above, the argument will produce a bounded *real*-linear functional ϕ such that $\phi(x) < c < \phi(b)$ for all $x \in A$. This ϕ is the real part of the bounded *complex*-linear functional

$$\psi(x) = \phi(x) - i\phi(ix)$$

(this is the same complexification trick that we used in the proof of the complex form of the Hahn-Banach theorem). Now define F_b to be the inverse image under ψ of the closed half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \leqslant c\}$ and complete the proof as before. □

Lecture 28

Applications of the Baire category theorem

Exercise 28.1. Give an example of a separately continuous map that is not jointly continuous.

Solution. The function (from $\mathbb{R}^2$ to $\mathbb{R}$)

$$(x,y) \mapsto f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is separately continuous. Indeed, if $y \neq 0$, $f(x,y)$ is clearly a continuous function of x, and if $y = 0$, $f(x,y) = 0$ identically. Similarly for x fixed as a function of y. However f is not jointly continuous, because any neighborhood of 0 contains points (ϵ, ϵ), for which $f(x,y) = \frac{1}{2}$, whereas $f(0,0) = 0$. ☐

Lecture 29

The open mapping and closed graph theorems

Exercise 29.1. Prove the above statements.

Solution. If $g \in L^2[0,1]$ is supported in the interval $[\epsilon, 1]$ for some $\epsilon > 0$, then the function $f(x) = g(x)/x$ is well-defined and belongs to L^2 with $\|f\| \leqslant \epsilon^{-1}\|g\|$. Thus g belongs to the range of T. But now for any $g \in L^2$ the functions $g_n = g\chi_{[1/n,1]}$ belong to the range of T and $g_n \to g$ in L^2 by the dominated convergence theorem. Thus the range of T is dense.

However, the constant function 1 is not in the range of T, since if $Tf = 1$ then $f(x) = 1/x$ almost everywhere, and the function $1/x$ is not in $L^2[0,1]$. Thus the range of T, being dense, cannot be closed (otherwise it would be all of $L^2[0,1]$). $\square$

Lecture 30

Operators on Hilbert Space

Exercise 30.1. Show that in the above situation the norm of M_f is exactly equal to the L^∞ norm of f.

Solution. We need to show that for any $\epsilon > 0$ there is $g \in L^2$ such that

$$\|M_f g\|_2 \geqslant (C - \epsilon)\|f\|_\infty \|g\|_2.$$

(The subscripts denote the norms in L^2 and L^∞.) By definition of the L^∞ norm, there is a subset $E \subseteq X$ of positive measure such that $|f(x)| \geqslant C - \epsilon$ for all $x \in E$. Since we are working with a σ-finite measure space, E contains a subset of positive *finite* measure; replace E by this subset and we may assume that E itself has finite measure. Let $g = \chi_E$. Then g belongs to L^2 with $\|g\|_2 = \mu(E)^{1/2}$ and

$$\|M_f g\|_2 = \left(\int_E |f|^2 d\mu\right)^{1/2} \geqslant (C - \epsilon)\mu(E)^{1/2}$$

as required. $\square$

Lecture 31

More about operators

Exercise 31.1. Use the closed graph theorem to show that if T is a linear map from a Hilbert space to itself, satisfying

$$\langle Tu, v\rangle = \langle u, Tv\rangle$$

for all $u, v \in H$, then T is an operator (i.e., it is bounded).

Solution. Suppose that $u_n \to u$ and $Tu_n \to w$. Then

$$\langle Tu, v\rangle = \langle u, Tv\rangle = \lim\langle u_n, Tv\rangle = \lim\langle Tu_n, v\rangle = \langle w, v\rangle$$

for every $v \in H$. Thus $\langle Tu - w, v\rangle = 0$ for every v, whence $Tu = w$. This proves that the graph of T is closed, so T is continuous by the closed graph theorem. This is a very early theorem (Hellinger-Toeplitz, around 1903). ☐

Lecture 32

The spectral theorem for compact operators

Exercise 32.1. Extend the two lemmas to *normal* operators T (those for which T commutes with T^*).

Solution. Let T be a normal operator. Then

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|^2$$

and thus $\ker(T) = \ker(T^*)$. Applying this to the normal operator $T - \lambda I$ instead of T gives $\ker(T - \lambda I) = \ker(T^* - \bar{\lambda} I)$. In other words, every eigenvector T (with eigenvalue λ) is also an eigenvector for T^* (with eigenvalue $\bar{\lambda}$).

Now let E be the eigenspace $\ker(T - \lambda I)$. If $x \in E^\perp$ and $y \in E$ then

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \bar{\lambda} y\rangle = 0.$$

Thus $Tx \in E^\perp$ also.

Let x_1, x_2 be eigenvectors for T with eigenvalues λ_1, λ_2. Then

$$\lambda_2\langle x_1, x_2\rangle = \langle x_1, Tx_2\rangle = \langle T^*x_1, x_2\rangle = \langle \bar{\lambda}_1, x_2\rangle = \lambda_1\langle x_1, x_2\rangle$$

so either $\lambda_1 = \lambda_2$ or $\langle x_1, x_2\rangle = 0$. ☐

Lecture 33

More about spectral theory

Exercise 33.1. Prove this result *without* using the spectral theorem. In fact, prove that any compact operator T is the limit of the sequence P_nT of finite rank operators, where P_n is the orthogonal projection onto the subspace spanned by the first n elements of some fixed orthonormal basis.

Solution. The key to the proof is that the projections P_n and $I - P_n$ are *orthogonal*, so have norm 1. Suppose that P_nT does *not* tend to T in norm. Then there exist $\epsilon > 0$ and a sequence $\{x_n\}$ of unit vectors such that $\|P_nTx_n - x_n\| > \epsilon$ for infinitely many n. Since T is compact, we may assume (by passing to a subsequence if necessary) that Tx_n is convergent, say to y and that $\|P_nTx_n - x_n\| > \epsilon$ for all n. But now write

$$P_nTx_n - Tx_n = (I - P_n)(Tx_n - y) + (P_ny - y).$$

Since $Tx_n \to y$ and $\|I - P_n\| \leqslant 1$, the first term here tends to 0. The second term also tends to 0 by the properties of orthonormal bases. This is a contradiction. □

Lecture 34

The Weierstrass Approximation Theorem

Exercise 34.1. In the complex case the Stone-Weierstrass theorem takes the following form: a $*$-subalgebra of $C(X)$ which contains the constants and separates points is dense, where the $*$-condition means that the algebra is closed under (pointwise) complex conjugation. Prove this.

Solution. Let A be a $*$-subalgebra of $C(X)$. If $f = u + iv$ with u and v real, then $u = \frac{1}{2}(f + \bar{f})$ and $v = \frac{1}{2i}(f - \bar{f})$. Therefore, u and v belong to A. It now follows that the collection B of all real and imaginary parts of members of A is a subset of A, closed under addition, and multiplication and containing the (real-valued) constant functions; in other words it is a subalgebra of $C_{\mathbb{R}}(X)$. Moreover, since A separates points, so does B (if $f(x) \neq f(y)$ then either $u(x) \neq u(y)$ or $v(x) \neq v(y)$). Thus B is dense in $C_{\mathbb{R}}(X)$ by the real form of the Stone-Weierstrass theorem, and it easily follows that A is dense in $C(X)$. ☐