Lecture 1

Extended Homework 1

Exercise 1.1. Let A and B be subsets of $\mathbb{R}$, with A compact and B closed. Show that the set $A+B$, defined by

$$A+B=\{a+b:a\in A,\ b\in B\}$$

is closed. Give an example to show that if A and B are both closed, $A+B$ need not be closed.

Exercise 1.2. Let X be a metric space. For a subset Y of X define $N_\epsilon(Y)=\bigcup_{y\in Y}B(y;\epsilon)$. Show that the intersection $\bigcap_{\epsilon>0}N_\epsilon(Y)$ is equal to $\overline{Y}$, the closure of Y.

Now suppose that Y and Y' are closed subsets of a compact metric space X. Define the *Hausdorff distance* $d^H(Y,Y')$ by

$$d^H(Y,Y')=\inf\{\epsilon>0:Y\subseteq N_\epsilon(Y'),Y'\subseteq N_\epsilon(Y)\}.$$

Show that d^H makes the collection of all closed subsets of X into a metric space.

(Extra credit) Show that this metric space is itself compact. (Show that it is complete and totally bounded.)

Exercise 1.3. Let $f\colon[0,1]\to\mathbb{R}$ be a function. One says that f is *absolutely continuous* if, for any $\epsilon>0$, there is $\delta>0$ with the following property: given any (finite or countable) collection $[a_k,b_k]$ of disjoint closed subintervals of $[0,1]$,

$$\text{if }\sum_k|a_k-b_k|<\delta,\quad\text{then }\sum_k|f(a_k)-f(b_k)|<\epsilon.$$

Prove that every absolutely continuous function is continuous. Show that the Cantor function is an example of a continuous function that is *not* absolutely continuous.

Exercise 1.4. A real number α is called a *Liouville number* if it is irrational and for each n there exist integers p,q with $q\geqslant 2$ such that

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^n}.$$

Show that the set L of Liouville numbers is residual (in particular, it is nonempty).

(Liouville showed that any Liouville number is *transcendental*, i.e. is not the root of any polynomial equation with integer coefficients. Can you give an explicit example of a Liouville number?)

Exercise 1.5. In the complete metric space $C_{\mathbb{R}}[0,1]$, let $A_{m,n}$, for $m,n \in \mathbb{N}$, be the set

$$\left\{ f\colon [0,1] \to \mathbb{R} :\ \exists x \in [0,1] \text{ s.t. } \left| \frac{f(x+h)-f(x)}{h} \right| \leqslant m \ \forall 0 < |h| \leqslant 1/n \right\}.$$

Show that $A_{m,n}$ is a closed set with empty interior. Deduce that there exist continuous functions $[0,1] \to \mathbb{R}$ that are not differentiable at any point. (Use the Baire category theorem).

Exercise 1.6. Let X be a compact metric space, and let M be the compact (in particular, complete) metric space of closed subsets of X, equipped with the Hausdorff distance (see problem 1). Suppose that $f_1, \ldots, f_n\colon X \to X$ are strict contractions. Prove that the map $f\colon M \to M$, defined by

$$f(K) = f_1(K) \cup \cdots \cup f_n(K)$$

is a strict contraction, and therefore that there is a unique closed $K \subseteq X$ such that $K = f_1(K) \cup \cdots \cup f_n(K)$. Taking $X = [0,1]$, find two contractions f_1, f_2 such that the unique K so defined is the Cantor set.

(This is the basic idea of Barnsley's method of *fractal image compression*. The functions $f_1, \ldots, f_n$ constitute an *iterated function system* encoding the set K. As the example of the Cantor set shows, a very simple IFS can encode a rather complicated set.)

Exercise 1.7. An infinite product $\prod_{n=1}^{\infty} a_n$ is said to converge if the partial products $\prod_{n=1}^{N} a_n$ tend to a limit as $N \to \infty$. Prove a version of Weierstrass' M-test for infinite products, namely: if $f_n(x)$ is a sequence of functions (on a metric space X, which you may assume is compact) and $|f_n(x) - 1| \leqslant M_n$, with $\sum M_n < \infty$, then the infinite product $\prod_{n=1}^{\infty} f_n(x)$ converges uniformly on X.

The gamma function is defined by the infinite product

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-z/n} \right\},$$

where γ is Euler's constant (the limit $\lim_{k\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log k)$.) Show that the infinite product converges uniformly on compact subsets of $\mathbb{C}$.

(Extra credit) Prove the identity $\Gamma(z+1) = z\Gamma(z)$.

Exercise 1.8. A metric space is said to be *separable* if it has a dense countable subset. For example, $\mathbb{R}$ is separable because $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

Let X be a compact metric space. Prove that X is separable. Prove further that $C(X)$ is separable. (You may find it helpful to use the Stone-Weierstrass theorem, though it is also possible to do without it.)