Lecture 8 The Homotopy Form of Cauchy's Theorem

(8.1) DEFINITION: *Let Ω be an open subset of $\mathbb{C}$ and let $\gamma_0, \gamma_1 \colon [0,1] \to \mathbb{C}$ be paths which have the same endpoints (i.e. $\gamma_0(0) = \gamma_1(0) = a$ say, and $\gamma_0(1) = \gamma_1(1) = b$ say). The paths are* homotopic *in Ω if there is a continuous map $\Gamma \colon [0,1] \times [0,1] \to \mathbb{C}$ with*

$$\gamma_0(t) = \Gamma(0,t),\ \gamma_1(t) = \Gamma(1,t), \quad a = \Gamma(s,0),\ b = \Gamma(s,1),$$

for all s, t, $0 \leqslant s, t \leqslant 1$.

The idea of this definition is that γ_0 can be continuously deformed within Ω into γ_1, keeping its endpoints fixed. Homotopy is an equivalence relation on the class of paths with given fixed endpoints.

(8.2) THEOREM: *Let Ω be an open subset of $\mathbb{C}$ and let f be holomorphic in Ω. Let γ_0 and γ_1 be homotopic in Ω. Then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

In particular if γ is closed and nullhomotopic *(homotopic to a constant path), then $\oint_\gamma f(z)\,dz = 0$.*

PROOF: The idea is to apply Cauchy's theorem for convex regions repeatedly.

For each $p \in \Omega$ there is a disk D_p (convex!) centered at p and contained in Ω (because Ω is open). The inverse images $\Gamma^{-1}(D_a)$ are open subsets of $I^2 = [0,1] \times [0,1]$, and they form a cover of I^2. Since I^2 is a compact metric space, Lebesgue's Covering Lemma gives us a number $\varepsilon > 0$ such that any subset of I^2 having diameter $< \varepsilon$ lies within some $\Gamma^{-1}(D_p)$.

Choose an integer $n > 2/\varepsilon$. Define, for $0 \leqslant j, k \leqslant n$,

$$z_{j,k} = \Gamma(j/n, k/n).$$

By construction, for any j, k the four points $z_{j,k}$, $z_{j+1,k}$, $z_{j+1,k+1}$, $z_{j,k+1}$ lie in a convex subset of Ω. Hence by Cauchy's theorem for convex regions

$$\int_{[z_{j,k}, z_{j+1,k}]} f(z)dz + \int_{[z_{j+1,k}, z_{j+1,k+1}]} f(z)dz \\ - \int_{[z_{j,k+1}, z_{j+1,k+1}]} f(z)dz - \int_{[z_{j,k}, z_{j,k+1}]} f(z)dz = 0.$$

Sum these equations over all j, k between 0 and $n-1$. The integrals over interior edges cancel. The integrals over top and bottom edges (in the usual picture of I^2) vanish because these are along constant paths. We are left therefore with integrals along the left and right hand edges of I^2. But these sum to $\int_{\gamma_0} f(z)dz$ and $\int_{\gamma_1} f(z)dz$ respectively. We conclude that

$$\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = 0$$

as asserted. ■

Here is an application. Let D be a disk and suppose that f is holomorphic on an open set Ω which contains the closure of D. Then for any point a in the interior of D,

$$f(a) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z-a} dz.$$

This is *Cauchy's integral formula*.

PROOF: We shall show that the absolute value of the difference between the two sides of the equation can be made less than any prescribed $\varepsilon > 0$.

Let ε be given. Because f is continuous at a there is δ such that $|f(z) - f(a)| < \varepsilon$ whenever $|z-a| \leqslant \delta$.

Let γ be a circular contour around a of radius δ. Then γ is homotopic, in $\Omega \setminus \{a\}$, to ∂D, and so

$$\oint_{\partial D} \frac{f(z)}{z-a} dz = \oint_{\gamma} \frac{f(z)}{z-a} dz$$

by Theorem 8.2 applied to the function $z \mapsto f(z)/(z-a)$, which is holomorphic in $\Omega \setminus \{a\}$.

But now

$$f(a) - \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z) - f(a)}{z-a} dz.$$

Thus by the standard estimate

$$\left| f(a) - \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} dz \right| \leqslant \frac{1}{2\pi} \cdot 2\pi\delta \cdot \frac{\varepsilon}{\delta} = \varepsilon.$$

We have shown that the difference between the two sides of Cauchy's integral formula is less than the arbitrarily prescribed ε. This is what was required. ■