Lecture 7   Cauchy's Theorem for a Convex Region

If a and b are complex numbers, the *line segment* $[a, b]$ is the path $t \mapsto (1-t)a + tb$, $0 \leqslant t \leqslant 1$; in other words the straight line from a to b traversed at constant speed.

(7.1) DEFINITION: *A subset Ω of $\mathbb{C}$ is* convex *if whenever $a, b \in \Omega$, all the points of $[a, b]$ (that is all the points $(1-t)a + tb$, $0 \leqslant t \leqslant 1$) lie in Ω as well.*

Examples: The whole plane $\mathbb{C}$ is convex. A disk is convex. A punctured disk (a disk minus its center point) is not convex.

(7.2) PROPOSITION: *Let Ω be a convex open subset of $\mathbb{C}$ and let f be a continuous function on Ω. The following properties of f are equivalent:*

- *(a) f has an antiderivative: there is a holomorphic function F in Ω with $F'(z) = f(z)$ for all $z \in \Omega$;*
- *(b) The integral $\oint_\gamma f(z)dz$ vanishes for all closed contours γ in Ω;*
- *(c) The integral $\oint_{\partial T} f(z)dz$ vanishes for all* triangular *contours ∂T in Ω.*

Remarks. According to Cauchy's theorem for a triangle (theorem 6.3 from last lecture), hypothesis (c) applies to holomorphic functions f on Ω. (To see this, you need to notice that convexity tells you that if the boundary of a triangle T is in Ω, so its the interior.) Thus holomorphic functions on Ω satisfy (a) and (b) as well — this is Cauchy's theorem for a convex region. Conversely it can be proved that any function satisfying (a) is holomorphic — so the hypotheses are exactly equivalent to f being holomorphic. But this proof needs Cauchy's integral formula, which we will prove in a couple of lectures.

PROOF: Clearly (a) implies (b) implies (c). Suppose (c). Fix a base point $z_0 \in \Omega$ and define

$$F(z) = \int_{[z_0, z]} f(w)\, dw.$$

We shall show that F is holomorphic and its derivative is f.

Note that

$$F(z+h) - F(z) = \int_{[z, z+h]} f(w) dw$$

by the hypothesis (c) applied to the triangle with vertices z_0, z, and $z + h$. Therefore

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z))dw;$$

notice that $f(z)$ is a constant as far as the w-integration is concerned. Given any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(w) - f(z)| < \varepsilon$ provided $|w - z| < \delta$; this is because f is continuous at z. Therefore, by the estimate from last time, if $|h| < \delta$ we have

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leqslant \frac{1}{h} \cdot \varepsilon \cdot h = \varepsilon.$$

It follows that

$$\frac{F(z+h) - F(z)}{h} \to f(z) \quad \text{as } h \to 0,$$

proving that $F'(z)$ exists and equals $f(z)$ as required. ■