Lecture 6 Simple Estimates; Cauchy for a Triangle

We will need some simple estimates for integrals. The basic one is

(6.1) PROPOSITION: *Let γ be a (piecewise differentiable) path (in $\mathbb{C}$) and let f be a function defined and continuous on the image of γ. Then*

$$\left|\int_\gamma f(z)\,dz\right| \leqslant M\,\mathrm{Length}(\gamma),$$

where M is an upper bound for $|f(z)|$ along γ.

PROOF: Use the definition

$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

By real-variable theory

$$\left|\int_\gamma f(z)\,dz\right| \leqslant M\int_0^1 |\gamma'(t)|dt = M\,\mathrm{Length}(\gamma)$$

as required. ■

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(6.2) COROLLARY: *If $\{f_n\}$ is a sequence of continuous functions that converges* uniformly *to the function f on the image of a path γ (of finite length!), then $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$.*

(6.3) THEOREM: *(Cauchy for a triangle) Let Ω be an open subset of $\mathbb{C}$, and let T be a triangle that is contained (interior and boundary) within Ω. Let ∂T denote the closed path obtained by traversing the three sides of T counterclockwise. For any holomorphic function f on Ω, one has*

$$\oint_\partial T f(z)\,dz = 0.$$

PROOF: Proof by contradiction, so suppose the result false. Define the 'badness' $B(T)$ of a triangle T to be

$$B(T) = \left|\oint_{\partial T} f(z)\,dz\right| \Big/ \mathrm{diam}(T)^2.$$

By hypothesis there is some triangle T with $B(T) > 0$, say $B(T) = 5\varepsilon$.

Subdivide $T = T_0$ into four sub-triangles by bisecting each side. Each sub-triangle has half the diameter of T. On the other hand, the integrals $\oint f(z)dz$ around the boundaries of the four sub-triangles total the integral around the boundary of T. Consequently, one such integral must have absolute value at least $\frac{1}{4}$ of the absolute value of the integral around the boundary of T. We conclude that one sub-triangle, call it T_1, has $B(T_1) \geqslant B(T)$.

Repeat this process obtaining a nested sequence of triangles $T_0 \supset T_1 \supset T_2 \supset \cdots$, each half the diameter of the preceding one, and all with badness $B(T_n) \geqslant B(T) = 5\varepsilon$.

A compactness argument shows that $\bigcap T_n = \{z_0\}$ for some point z_0.

Now since f is differentiable at z_0 we have

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + e(z),$$

with a 'small' error term $e(z)$; in particular, there is $\delta > 0$ such that if $|z - z_0| < \delta$ then $|e(z)| < \varepsilon|z - z_0|$.

The function

$$z \mapsto f(z_0) + (z - z_0)f'(z_0)$$

is linear and thus has an antiderivative; so its integral around any closed contour is zero. Consequently

$$\oint_{\partial T_n} f(z)\,dz = \oint_{\partial T_n} e(z)dz$$

for each triangle T_n. Pick n large enough that the diameter d of T_n is less than δ. Then by the estimate 6.1,

$$\left|\oint_{\partial T_n} e(z)dz\right| \leqslant \varepsilon d \cdot 3d = 3d^2\varepsilon,$$

so $B(T_n) \leqslant 3\varepsilon$ and this is a contradiction. ■