Lecture 3 Examples of Holomorphic Functions

The usual rules of differentiation (sum rule, product rule, quotient rule, chain rule) are valid with the same proofs. Thus polynomials are holomorphic on teh whoel complex plans ('entire functions'); rational functions are holomorphic except where the denominator vanishes (a finite set of points).

Most limit processes preserve holomorphicity. In fact

(0.1) THEOREM: *The limit of a locally uniformly convergent sequence of holomorphic functions is holomorphic.*

We will prove this later; it requires some technique. A consequence is

(0.2) PROPOSITION: *A power series in the complex plane converges to a holomorphic function (within its circle of convergence).*

PROOF: As remarked, this follows from general results, but here is a bare hands proof. Suppose that $f(z) = \sum a_n z^n$, where the power series converges in the unit disc. Consequently, $|a_n|(1-\delta)^n$ is bounded for all $\delta > 0$, and then it follows by simple estimates that $\sum n|a_n|(1-\delta)^n$ converges for all $\delta > 0$. Thus, the power series

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

converges in the unit disc; we will prove that f is differentiable there with derivative g.

Consider

$$\frac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} a_n \frac{(z+h)^n - z^n}{h} = \sum_{n=0}^{\infty} a_n \sum_{j=0}^{n-1} (z+h)^{n-j} z^j.$$

Thus

$$\frac{f(z+h)-f(z)}{h} - g(z) = \sum_{n=0}^{\infty} a_n \sum_{j=0}^{n-1} ((z+h)^{n-j} - z^{n-j}) z^j.$$

Let $1 - |z| = 2\delta$. Then for $|h| < \delta$ the series on the right is absolutely convergent, uniformly in h; and the individual summands tend to zero as $h \to 0$. It follows that

$$\frac{f(z+h)-f(z)}{h} - g(z) \to 0 \quad \text{as } h \to 0,$$

which shows that $f'(z)$ exists and equals $g(z)$. ■

Many examples of holomorphic functions follow: e.g. e^z, $\sin z$, $\cos z$ (defined by the usual power series). Notice that $e^{\bar{z}} = \overline{e^z}$, because the power series has real coefficients, and so

$$|e^z|^2 = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2\Re z},$$

so that $|e^z| = e^{\Re z}$. Note also the famous formula (De Moivre)

$$e^{iz} = \cos z + i \sin z.$$

There are lots of other cute functions.

EXAMPLE: The gamma function. This is defined for $\Re z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$$

Integration by parts gives the recurrence formula

$$\Gamma(z+1) = z\Gamma(z).$$

By induction this tells us that $\Gamma(n+1) = n!$. Moreover, Γ is holomorphic in z. The formula

$$\Gamma(z) = z^{-1}\Gamma(z+1)$$

allows us to extend Γ inductively to a function on the whole complex plane with singularities at $0, -1, -2, \ldots$. This is an example of *analytic continuation*.

Gamma satisfies some neat identities. Here are a few:

(a) (Euler's Reflection Formula) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$;

(b) (Legendre's Duplication Formula) $\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma(z + \frac{1}{2})$;

(c) (Stirling's Formula) $\Gamma(x) \asymp \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}$.