

Lecture 22 Hyperbolic Geometry

(21.1) PROPOSITION: *Every biholomorphic automorphism of the disk $D(0; 1)$ is of the form*

$$z \mapsto e^{i\varphi} \frac{z - a}{1 - \bar{a}z}$$

for some a and φ .

PROOF: It is trivial to check that the automorphisms of the form given above form a group G which acts transitively on the disk. Let f be any automorphism; we want to prove $f \in G$. Since G acts transitively we may compose with an element of G , if necessary, to arrange that $f(0) = 0$. Now by Schwarz' lemma, $|f(z)| \leq |z|$. Moreover, by Schwarz' lemma applied to f^{-1} , $|z| \leq |f(z)|$. Thus equality holds in Schwarz' lemma and f is a rotation. ■

It is a corollary to this that the group G of automorphisms of the disk preserves a certain geometric structure.

(21.2) DEFINITION: *The hyperbolic distance between two points of $D(0; 1)$ is*

$$d(a, b) = 2 \tanh^{-1} \left| \frac{b - a}{1 - \bar{a}b} \right|.$$

By construction, $d(a, b)$ is a positive symmetric function of a and b which vanishes only if $a = b$. Moreover, we may say that $d(a, b) = 2 \tanh^{-1} |g(b)|$, where g is any element of G that maps a to 0. This makes it obvious that $d(a, b) = d(h(a), h(b))$ for any $h \in G$, ie d is *invariant* under the automorphism group G . The reason for the hyperbolic tangent is to make the addition law for distances hold: if $a, 0, b$ are collinear along the real axis (say $a < 0$ and $b > 0$) then $d(a, b) = d(a, 0) + d(0, b)$. This follows from the addition law

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

It is a homework exercise to prove that d satisfies the triangle inequality, so it is a distance function (metric) on the disk.

The distance function d defines a *non-euclidean geometry* in the unit disk, called *hyperbolic geometry*. In this geometry, the 'lines' (shortest paths) are circular arcs or line segments that meet the unit circle at right angles. It can be shown that this new geometry satisfies all the axioms of Euclid except that the

parallel postulate does not hold: given a line, and a point not on that line, one can find infinitely many lines through the given point that do not meet the given line.

It was noted by Pick that Schwarz' lemma has a beautiful invariant statement in terms of hyperbolic geometry, namely:

(21.3) PROPOSITION: *Any holomorphic map from the unit disk to itself must decrease all hyperbolic distances.*

PROOF: If such a map sends 0 to 0, this is a consequence of the usual form of Schwarz' lemma. Since G acts transitively and preserves the hyperbolic distance, the general case can be reduced to that special one by composing with a suitable element of G . ■

The *upper half plane* $H = \{z : \Im(z) > 0\}$ is biholomorphic to the disk: the Möbius transformation $z \mapsto (z - i)/(z + i)$ maps H biholomorphically onto $D(0; 1)$. Thus, we can find the group of automorphisms of H from our knowledge of the group of automorphisms of $D(0; 1)$. The result is:

(21.4) PROPOSITION: *Every automorphism of H is of the form $z \mapsto (az + b)/(cz + d)$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.*

One says that the automorphism group of H is the *projective special linear group* $PSL(2, \mathbb{R})$.