Lecture 2 Complex Differentiability

The matrix of multiplication by $z = x + iy = re^{i\theta}$, considered as a linear transformation of $\mathbb{R}^2$, is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \quad \text{or} \quad r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Moreover we have

(0.1) LEMMA: *Every conformal (angle-preserving) linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, with positive determinant, is of this sort (i.e. is multiplication by a complex number).*

PROOF: Express the conformality condition in terms of dot products. ■

Now we begin to do some analysis. Recall that the modulus $|z|$ of a complex number z satisfies the *triangle inequalities*

$$\left|\,|z| - |w|\,\right| \leqslant |z \pm w| \leqslant |z| + |w|.$$

These inequalities allow us to carry over all the elementary theory of limits to the complex case, with the same proofs. (Formally, $\mathbb{C}$ becomes a metric space with distance $d(z, w) = |z - w|$.)

(0.2) DEFINITION: *Let $\Omega \subseteq \mathbb{C}$ be open. A function $f\colon \Omega \to \mathbb{C}$ is* differentiable *at $a \in \Omega$ if there exists a complex number (denoted by $f'(a)$) such that*

$$f(a + h) = f(a) + f'(a)h + o(|h|)$$

as $h \to 0$.

Here $o(|h|)$ denotes a term that tends to zero faster than $|h|$ does, as $h \to 0$. Since $\mathbb{C}$ is a field we may also divide and rewrite in the equivalent form

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}.$$

We say f is *holomorphic* in Ω if it is differentiable at all points of Ω.

How does this differ from *real-variable* differentiability? Recall that $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at a in the real-variable sense if there is a 2×2 matrix $Dg(a)$ such that

$$g(a + h) = g(a) + (Dg(a)) \cdot h + o(|h|).$$

Comparing definitions we see that the 'extra ingredient' for complex differentiability is that $Dg(a)$ should be the matrix of multiplication by a complex number, i.e. should be of the form given in the lemma above. Since the components of the matrix Dg are the partial derivatives of the components of g, we see that $f = u+iv$ is complex-differentiable at a iff it is real-differentiable and the partials of u and v at a satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$