Lecture 16 The Gamma Function

We will develop the properties of the Γ function. We begin with the infinite product expansion.

(16.1) DEFINITION: Euler's constant γ *is the limit* $\lim_{n\to\infty}\big(\sum_{k=1}^{n} k^{-1} - \log n\big)$.

Since $\frac{1}{n} \leqslant \log(n) - \log(n-1) \leqslant \frac{1}{n-1}$ one easily checks that the expression under the limit is decreasing in n and bounded below, so convergent. The numerical value of Euler's constant is approximately $0 \cdot 577215665$.

(16.2) PROPOSITION: *The* Γ *function* $\Gamma(z)$ *satisfies*

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n}\right].$$

The infinite product converges to an entire function on $\mathbb{C}$. *In particular,* $\Gamma(z)$ *never vanishes.*

The convergence of an infinite product is defined in the obvious way — as convergence of the partial products. It is easy to check that $\prod(1 + a_n)$ converges if $\sum |a_n|$ converges. This can be used to show that the product above converges. The convergence is locally uniform hence to an analytic function.

PROOF: Because of the uniqueness of analytic continuation it suffices to check the result for z real and positive. From homework we have the limit formula

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n!n^z}.$$

Rewrite the expression under the limit as

$$ze^{-z\log n} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) = ze^{z(\sum_{k=1}^{n} \frac{1}{k} - \log n)} \prod_{k=1}^{n} \left[\left(1 + \frac{z}{k}\right) e^{-z/k}\right].$$

Letting $n \to \infty$ we obtain the result. ■

(16.3) REMARK: Taking the log of the infinite product gives an infinite series for $\log\Gamma(x)$, x real. Differentiating we find that

$$\frac{d^2 \log\Gamma(x)}{dx^2} = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} \geqslant 0.$$

Thus Γ is *log-convex*. It can be proved that it's the *only* log-convex function which has $\Gamma(1) = 1$ and $x\Gamma(x) = \Gamma(x+1)$ (*Bohr-Mollerup theorem*).

Now the reflection formula.

(16.4) THEOREM: *We have* $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.

PROOF: By analytic continuation it is enough to check this for $z = a$, $0 < a < 1$. Write

$$\Gamma(1-a)\Gamma(a) = \int_{s=0}^{\infty}\int_{t=0}^{\infty} s^{-a}t^{a-1}e^{-(s+t)}dtds.$$

In the t-integral substitute $t = us$ to obtain

$$\int_{s=0}^{\infty}\int_{u=0}^{\infty} u^{a-1}e^{-s(1+u)}duds.$$

Reverse the order of integration and perform the inner s-integral to obtain

$$\int_{u=0}^{\infty} \frac{u^{a-1}}{1+u}du.$$

The substitution $u = e^t$ gives

$$\int_{-\infty}^{\infty} \frac{e^{at}\,dt}{1+e^t} = \frac{\pi}{\sin \pi z},$$

as we worked out by contour integration. ■

This has many interesting consequences. For instance, writing it as $\sin \pi z = -\pi/(z\Gamma(z)\Gamma(-z))$ and combining with the infinite product for $1/\Gamma(z)$ we get

$$\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

By logarithmic differentiation (valid for real z, and then analytic continuation takes care of the rest)

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{z-k}.$$

This last result can be used to find the value of Riemann's zeta function at even integers. Recall that the *Bernoulli numbers* are defined by $z/(e^z - 1) = \sum B_n z^n/n!$; we have $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and so on (the odd ones vanish after B_1). Then we have

(16.5) PROPOSITION: *For each even integer* $2m$,

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{(-1)^{m+1} 2^{2m-1} B_{2m} \pi^{2m}}{(2m)!}.$$

PROOF: Write

$$z \cot z = iz \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = iz + \frac{2iz}{e^{2iz} - 1} = 1 - \sum_{m=1}^{\infty} (-1)^{m+1} B_{2m} \frac{2^{2m} z^{2m}}{(2m)!}.$$

On the other hand, using our infinite sum for the cotangent,

$$z \cot z = 1 + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2 \pi^2}.$$

Expand the summand in a binomial series and rearrange as a power series in z (justified by absolute convergence). The coefficient of z^{2m} is

$$\sum_{n=1}^{\infty} \frac{2}{\pi^{2m} n^{2m}}.$$

Comparing coefficients with those appearing in the previous expansion of $z \cot z$ (justified by uniqueness of Taylor coefficients) we get the result. ■