

Lecture 12 Consequences of the Residue Theorem

We can use integration to count zeroes. Let f be holomorphic in Ω and not identically zero. Then for any compact subset K of Ω , the set of zeroes of f in K is finite (by the principle of isolated zeroes). The number of zeroes of f in K is by definition the sum of the orders of these zeroes. In other words we count zeroes ‘with multiplicity’: a double zero counts as two simple zeroes.

(12.1) PROPOSITION: *(Zero counting) Let f be holomorphic and not identically zero in a connected open set Ω , and let γ be a Jordan curve with $\text{Int}(\gamma) \subseteq \Omega$; suppose that f does not vanish on ∂D . Then the number of zeroes of f inside D (counted with multiplicity as above) equals*

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z) dz}{f(z)};$$

that is the winding number $n(f \circ \gamma; 0)$ of the path $f \circ \gamma$ about the origin.

Obvious corollary: $n(f \circ \gamma; a)$ counts the number of times f takes the value a .

PROOF: The function f'/f is holomorphic except at the zeroes of f .

Near a zero of order n , put $f(z) = (z - a)^n g(z)$, where g does not vanish at a . Then $f'(z) = n(z - a)^{n-1} g(z) + (z - a)^n g'(z)$ and so

$$\frac{f'(z)}{f(z)} = \frac{n}{z - a} + \frac{g'(z)}{g(z)}.$$

The second term is holomorphic at a so it has no residue. Thus the residue of f'/f at a is equal to the residue of the first term, that is n , the order of the zero. Now the result follows from the Residue Theorem. ■

Addendum: If f is meromorphic, and we count a pole of order n as z zero of order $-n$, the result still holds to count the total of poles and zeroes; the proof is the same.

(12.2) COROLLARY: *A non-constant holomorphic map on a connected open set Ω is an open map; it takes open subsets (of Ω) to open subsets of \mathbb{C} .*

PROOF: It is enough to show that, given any $a \in \Omega$, there is a neighborhood W of a in Ω such that $f(W)$ is a neighborhood of $b = f(a)$.

Since f is non-constant there is a disk $D(a; \varepsilon)$ on which $f(z)$ does not take the value b except at $z = a$. Let γ be a circular contour around a of radius $\varepsilon/2$. Then $n(f \circ \gamma; b) > 0$ and so (by continuity) $n(f \circ \gamma; b') > 0$ for b' close enough to b , say $|b' - b| < \delta$. Thus f takes the value b' within γ . But this is to say that if $W = D(a; \varepsilon/2)$, then $f(W)$ contains $D(b; \delta)$, and so is a neighborhood of b , as required. ■

(12.3) COROLLARY: *A holomorphic map is locally injective at each point a for which $f'(a) \neq 0$; in other words, there is a neighborhood of a that is mapped one-to-one into \mathbb{C} by f .*

PROOF: Argue as in the previous corollary. Now $n(f \circ \gamma; b) = 1$ (since $f(z) - b$ has a simple zero at a) and so $n(f \circ \gamma; b') = 1$ also for all b' belonging to some disk $D(b; \delta)$. Let $U = f^{-1}(D(b; \delta)) \cap D(a; \varepsilon/2)$; then U is open and contains a , and f takes each value in $D(b; \delta)$ precisely once on U — in other words, f is injective on U . ■

(12.4) COROLLARY: *(Rouché's Theorem) Let f and g be holomorphic in an open set Ω which contains a Jordan curve γ and its interior. Suppose that neither f nor g has a zero on γ . If $|f(z) - g(z)| < |f(z)|$ for all $z \in \text{Im}(\gamma)$, then f and g have the same number of zeroes inside γ .*

PROOF: We claim that the paths $f \circ \gamma$ and $g \circ \gamma$ are homotopic in $\mathbb{C} \setminus \{0\}$. Indeed, the 'linear homotopy'

$$\Gamma(s, t) = (1 - s)f \circ \gamma(t) + sg \circ \gamma(t)$$

does the job. The only thing to check is that it never passes through zero. But putting $z = \gamma(t)$,

$$|\Gamma(s, t)| = |f(z) + s(g(z) - f(z))| \geq |f(z)| - s|f(z) - g(z)| > 0$$

so we're okay. ■

We can use this to give another proof of the Fundamental Theorem of Algebra.

(12.5) COROLLARY: *A polynomial equation of degree n must have n roots.*

PROOF: In Rouché's theorem, let $g(z) = a_0 + \cdots + a_n z^n$ be our polynomial equation and let $f(z) = a_n z^n$. Let γ be a circle, center the origin. If the radius of γ is large then $|f - g| \leq |f|$ on γ . Therefore, f and g have the same number of roots inside γ . But f obviously has n roots counted with multiplicity (namely, an n -tuple root at the origin); so g does too. ■