Lecture 10 The Residue Theorem

Topological preliminaries: For this discussion, recall the distinction between a *curve* γ (a mapping from an interval into $\mathbb{C}$) and its *image* $\mathrm{Im}(\gamma)$ (a subset of $\mathbb{C}$.) A curve $\gamma\colon [a,b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$.

(11.1) DEFINITION: *We will say that a closed curve γ is a* Jordan curve *if*

- *(a)* $\mathbb{C} \setminus \mathrm{Im}(\gamma)$ *has exactly two connected components, one unbounded and one bounded; we denote the bounded one by* $\mathrm{Int}(\gamma)$*;*
- *(b) For each $a \in \mathrm{Int}(\gamma)$, the closed curve γ is homotopic, in $\overline{\mathrm{Int}(\gamma)} \setminus \{a\}$, to a small positively oriented circle around a.*

Notice that (b) implies in particular that $n(\gamma, a) = 1$ for each $a \in \mathrm{Int}(\gamma)$.

This definition is modeled on the properties of the circle and the disk that we used to prove the Cauchy integral formula. In fact, the Cauchy integral formula now holds for any Jordan curve, with the same proof.

(11.2) REMARK: It is a deep topological fact that any *simple* closed curve (that is, one which is injective as a map from the circle S^1 to $\mathbb{C}$ — it does not cross itself) is a Jordan curve. But we won't need this — every Jordan curve that we will have occasion to use can be easily verified to satisfy all the conditions of Definition 11.1.

Now discuss functions with singularities. Let a be a point of $\mathbb{C}$ and let f be a function holomorphic in $D(a;\varepsilon) \setminus \{a\}$, for some $\varepsilon > 0$. f is said to have an *isolated singularity* at a. We classify these into three types:

- (a) f has a *removable singularity* at a if it extends to a function holomorphic on $D(a;\varepsilon)$ (unique! — by uniqueness of analytic continuation).
- (b) f has a *pole* if $(z-a)^N f(z)$ extends to a function holomorphic on $D(a;\varepsilon)$ (for some N; the least N that will do is called the *order* of the pole.)
- (c) f has an *essential singularity* otherwise.

(11.3) DEFINITION: *The* residue *of a function f with an isolated singularity at a, written $\mathrm{Res}(f;a)$, is the value of the integral $(2\pi i)^{-1} \oint f(z)dz$, taken around a small circle around a.*

Let Ω be an open set. If f is holomorphic in Ω except for a set of isolated singularities, all of which are poles, we say that f is *meromorphic* in Ω.

(11.4) THEOREM: *Let f be meromorphic in an open set Ω. Let γ be a Jordan curve such that $\overline{\mathrm{Int}(\gamma)} \subseteq \Omega$. Assume that there are no singularities of f lying on γ. Then*

$$\oint_\gamma f(z)dz = \sum_{a \in \mathrm{Int}(\gamma)} \mathrm{Res}(f; a),$$

where the sum is taken over all poles a of f that lie inside γ.

PROOF: Induction on the total number, n, of poles of f lying inside γ (the number is finite because the set of poles consists entirely of isolated points and $\overline{\mathrm{Int}(\gamma)}$ is compact.)

For the base step of the induction ($n = 1$), suppose f has a single pole at a. By properties of Jordan curves, γ is homotopic in $\overline{\mathrm{Int}(\gamma)} \setminus \{a\}$ to a small circle around a. So, by Cauchy's theorem, the integral of $f(z)dz$ about γ is equal to the integral about such a small circle; that is, to $2\pi i\,\mathrm{Res}(f; a)$.

For the induction step, select a pole a and suppose that near a,

$$f(z) = (z-a)^{-N} g(z) = (z-a)^{-N} \sum_{k=0}^{\infty} c_k (z-a)^k,$$

using Taylor's theorem. Put $h(z) = (z-a)^{-N} \sum_{k=0}^{N} c_k (z-a)^k$. Then $h(z)$ is holomorphic everywhere except for a pole at a, and $f(z) - h(z)$ has the same singularities as f except that the pole at a has been removed; in particular it has $(n-1)$ poles. Now write

$$f(z) = h(z) + (f(z) - h(z)),$$

and apply the inductive hypothesis to both summands; then add the results. ■