Lecture 10 More Consequences of Cauchy's Theorem

(10.1) PROPOSITION: *(Removability of Singularities) Let f be continuous in an open set Ω and holomorphic in Ω except possibly at one point a. Then f is, in fact, holomorphic at a too.*

PROOF: Show that $\oint_{\partial T} f(z)dz = 0$ for all triangles — if the triangle surrounds a, make a homotopy to a very small circular path around a and use the basic estimate to show the integral around such a path is arbitrarily small. Then apply Morera's theorem. ■

(10.2) LEMMA: *Let f be holomorphic in a disc $D(a;r)$, but not identically zero there. Then there is a smaller disc $D(a;\rho)$ within which $f(z) = 0$ has no solution except possibly $z = a$.*

PROOF: Expand f in a Taylor series $f(z) = \sum c_n(z-a)^n$ about $z = a$. Since f does not vanish identically, some Taylor coefficient c_n is nonzero; let c_N be the first such coefficient (so that $c_N \neq 0$ but $c_k = 0$ for $k < N$). Then

$$f(z) = \sum_{n=N}^{\infty} c_n(z-a)^n = (z-a)^N(c_N + (z-a)g(z)),$$

where $g(z) = \sum_{k=0}^{\infty} c_{k+N+1}(z-a)^k$ is continuous at a. It follows that $|(z-a)g(z)| < \frac{1}{2}|c_N|$ for z sufficiently close to a, and thus $c_N + (z-a)g(z)$ does not vanish for z sufficiently close to a. ■

Terminology: A *zero* of f is a point where $f(z) = 0$. The exponent of the first nonvanishing term in the Taylor series of f about a zero (i.e. the number N above) is the *order* of the zero.

Let S be a subset of $\mathbb{C}$. An *accumulation point* of S is a point $a \in \mathbb{C}$ such that every neighborhood of a contains infinitely many members of S.

(10.3) PROPOSITION: *(The principle of isolated zeroes) Let $\Omega \subseteq \mathbb{C}$ be open and connected. If $f\colon \Omega \to \mathbb{C}$ is holomorphic and not identically zero, then the set of zeroes of f has no accumulation points in Ω.*

PROOF: Let $A \subseteq \Omega$ be the set of accumulation points of zeroes of f. Then A is closed. But by lemma 10.2, if $a \in A$ then some disc $D(a;\rho)$ is also contained in A. Thus A is open. Connectedness now gives that A is either empty or equal to Ω. ■

It follows that if f and g are holomorphic functions on Ω, and $f(z) = g(z)$ for all z belonging to some set having a limit point in Ω, then $f = g$ everywhere; this is the *uniqueness of analytic continuation*. For example, supposing that we know that a trig identity like

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

holds for all *real* z, w; we deduce that it holds for complex z, w too.

(10.4) LEMMA: *Let f be holomorphic in a disc $D(a; r)$ and not constant there. Then there are points $z \in D$ arbitrarily close to a at which $|f(z)| > |f(a)|$.*

PROOF: Take $a = 0$ wlog. Using the Taylor series write

$$f(z) = c_0 + \sum_{n=N}^{\infty} c_n z^n = c_0 + z^N(c_N + zg(z))$$

with g continuous, $c_N \neq 0$. By replacing $f(z)$ by $e^{i\varphi} f(e^{i\psi} z)$ for suitable constants φ, ψ we may assume wlog that c_0 and c_N are real and positive. But now for $x > 0$ real and small enough that $|xg(x)| < \frac{1}{2}|c_N|$,

$$\Re f(x) \geqslant c_0 + \tfrac{1}{2} c_N x^N > c_0$$

and so $|f(x)| > c_0 = |f(0)|$ also. ■

(10.5) THEOREM: *(Maximum principle) Let f be a nonconstant holomorphic function on a connected open set Ω. Then $|f|$ does not attain a local maximum (even nonstrict) anywhere on Ω.*

PROOF: If it did, then according to Lemma 10.4 it would be constant on a disk around the maximum point, hence everywhere in Ω by the uniqueness of analytic continuation. ■

The *minimum principle* states that $|f|$ attains a local minimum only at the zeroes of f. To prove it, apply the maximum principle to $1/f$.

(10.6) COROLLARY: *('Fundamental theorem of algebra') Every polynomial $p(z)$ has a root in $\mathbb{C}$.*

PROOF: A polynomial is holomorphic and has the property that $|p(z)| \to \infty$ as $|z| \to \infty$. Thus $|p(z)|$ must attain a (global!) minimum somewhere. By the minimum principle, the point where it does so is a root. ■

is greater than c_0