Lecture 1 Complex Is Simpler

Complex numbers $a + bi$, $a, b \in R$, are added and subtracted according to obvious rules. Multiplied using the rule $i^2 = -1$. We identify $a \in \mathbb{R}$ with $a + 0i \in \mathbb{C}$. Then $\mathbb{C}$ is a *commutative ring*.

In fact $\mathbb{C}$ is a *field*, ie we can divide by anything nonzero. If $z = a + bi$ is nonzero, then its multiplicative inverse is

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

as direct computation confirms. We write this as $z^{-1} = \bar{z}/|z|^2$.

$\mathbb{C}$ can be thought of as a plane. The Cartesian representation of a point in this plane is convenient for addition and substraction. For multiplication, the polar representation $z = r(\cos\theta + i \sin\theta) = re^{i\theta}$ is preferable. We have $|z| = r$. Notice that $e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$ (trig identities). Thus, to ultiply two complex numbers in polar form, multiply their moduli and add their *arguments* (the numbers θ). Multiplication by a fixed z acts on the plane by a stretch and a rotation, and it's therefore *conformal* (angle-preserving).

We are going to be studying *functions of a complex variable*, that is differentiable functions $f \colon \Omega \to \mathbb{C}$, where Ω is an open subset of the complex plane $\mathbb{C}$.

Why?

- Many such functions: exponential, sine, cosine, plus the 'higher transcendental functions' such as $\Gamma(z)$, $\zeta(s)$, $\wp(z)$ and so on — functions which are useful everywhere.

- Mystical connections: complex-differentiable (*holomorphic*) functions are very rigid — every part of them is connected to every other part.

Example related to circle of convergence. A *power series* like

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

or

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$

or

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

has a *radius of convergence* r such that the series converges for $|x| < r$ and diverges for $|x| > r$. For the series shown the radii of convergence are respectively $1, 1, \infty$.

When $|x|$ is smaller than the radius of convergence the series defines a function. For the examples shown here the functions defined are

$$\frac{1}{1+x}, \quad \frac{1}{1+x^2}, \quad e^x.$$

We now ask: What goes wrong at the radius of convergence? Why can't the series converge better? For example 1 the answer is clear; the function $f(x) = 1/(1+x)$ explodes at $x = -1$, so we can't hope to represent it by anything well-behaved (such as a power series) in a domain that includes that point. The biggest interval around 0 on which a power series for that function could possibly converge at all therefore has radius one; and, by golly, the power series *does* converge on that interval!

This is very satisfactory as an explanation of the misbehavior of the power series for $f(x) = 1/(1+x)$; but it fails to account for the misbehavior of the power series for $g(x) = 1/(1+x^2)$. Considered as a function of a *real variable* this is nice and smooth from $-\infty$ to ∞. Only when we look at it as a function of a complex variable do we see the singularities at $\pm i$ which prevent the power series converging on a disc of radius larger than 1.