Math 501 Homework 9 Due Monday, November 17th

(1) Let V and W be normed vector spaces. Show that a linear map $T\colon V \to W$ is continuous if and only if it is *bounded*, which means that there is a constant c with

$$\|T(v)\| \leqslant c\|v\|$$

for every $v \in V$.

Let T be continuous and let $U = \ker T$. Show that U is a closed subspace of V and that the expression

$$\|[v]\| = \inf\{\|u + v\| : u \in U\}$$

(where $[v]$ denotes the coset $v + U$, considered as an element of the quotient space V/U) defines a norm on V/U.

Show further that the map $[v] \mapsto T(v)$ is a well-defined continuous linear map from V/U to W.

(2) Let V be a (complex) normed vector space. A linear map $\varphi\colon V \to \mathbb{C}$ is called a *linear functional* on V. Show that a linear functional φ is continuous if and only if $\ker\varphi$ is closed. In fact, show that

$$|\varphi(v)| \leqslant \frac{1}{\varepsilon}\|v\|,$$

where ε is the distance from $\ker\varphi$ to a vector v_1 with $\varphi(v_1) = 1$.

Deduce, by induction on the dimension n, that if V is a finite-dimensional subspace of a normed vector space W, then V is closed in W and the linear isomorphism $V \to \mathbb{C}^n$ given by choosing a basis $\{v_1, \ldots, v_n\}$ of V is also a homeomorphism.

(3) Let V be a normed vector space. Suppose that the closed unit ball of V can be covered by finitely many balls $B(x_k; \frac{1}{2})$, $k = 1, \ldots, n$, of radius $\frac{1}{2}$. Show that the $\{x_k\}$ span V. (First use question 2 to show that the span of the $\{x_k\}$ is a closed subspace, then show that it contains the unit ball of V.)

Deduce that every locally compact normed vector space is finite-dimensional.