Math 501 Homework 7 Due Monday, November 3rd

(1) Let a be a point in the unit disk and let $d = d(0, a)$ be the hyperbolic distance from a to the origin. Show that

$$\cosh d = \frac{1 + |a|^2}{1 - |a|^2}.$$

Now consider a hyperbolic triangle ABC whose sides have hyperbolic length AB, BC, CA and whose angles are γ, α, β. Prove that

$$\cosh AB = \cosh AC \cosh BC - \sinh AC \sinh BC \cos \gamma.$$

(You will find it helpful to make a preliminary conformal transformation so that C lies at the origin.)

Deduce that $AB \leqslant AC + BC$ (the triangle inequality for hyperbolic distance).

(2) Suppose that the triangle in question 1 is small. Show that the ordinary (Euclidean) form of the cosine rule can be recovered from the hyperbolic cosine rule in question 1 by forming the Taylor expansion and neglecting all terms of higher than the second order in small quantities.

(3) Let u be a harmonic function on the plane. Suppose that $u(re^{i\theta})$ is less than 0 when $r = 1$, and is less than 1 when $r = R$. Show that

$$u(re^{i\theta}) \leqslant \frac{\log r}{\log R}$$

for all r between 1 and R. (Use the maximum principle.)