Math 501 Homework 12 Due Wednesday, December 10th

(1) State the spectral radius formula.

Let $H = L^2[0,1]$, and let V be the operator on H defined by

$$(Vf)(x) = \int_0^x f(t)\,dt.$$

Prove that

$$(V^k f)(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)\,dt$$

and hence calculate the spectrum of V.

(2) Let T be a *unitary* operator on a Hilbert space H; that is, $T^*T = I = TT^*$. Let K denote the kernel of $I - T$, and let L denote the closure of the range of $I - T$. Show that K and L are orthogonal complements, so that there is a direct sum decomposition $H = K \oplus L$.

Now let P denote the orthogonal projection onto K. Show that for each fixed $u \in H$,

$$\frac{1}{n+1}\sum_{j=0}^{n} T^j u \to Pu$$

as $n \to \infty$. (This is the *mean ergodic theorem*.)

(3) Let A be a compact self-adjoint operator on a Hilbert space H. Suppose that for all $x \in H$, $\langle Ax, x\rangle \geqslant 0$. Prove that there is a unique compact self-adjoint operator B such that $\langle Bx, x\rangle \geqslant 0$ for all x and $B^2 = A$.

Now let A be the operator on ℓ^2 defined by

$$A(x_1, x_2, x_3, \cdots) = (x_2/2, x_3/3, x_4/4, \cdots).$$

Prove that A is compact, but that there is no operator B such that $B^2 = A$.