Math 501 Homework 10 Due Monday, November 24th

(1) Find an inner product on the space of continuously differentiable complex-valued functions on $[-1,1]$ such that the corresponding norm is given by

$$\|f\| = \left\{ \int_{-1}^{1} (|x||f(x)|^2 + 3|f'(x)|^2)\, \mathrm{d}x \right\}^{\frac{1}{2}}.$$

Hence prove that, for any such function f,

$$\left| \int_{-1}^{1} (|x|^3 f(x) + 6x f'(x))\, \mathrm{d}x \right| \leqslant \frac{5}{\sqrt{3}} \left\{ \int_{-1}^{1} (|x||f(x)|^2 + 3|f'(x)|^2)\, \mathrm{d}x \right\}^{\frac{1}{2}}.$$

(2) Let $f\colon [-\pi,\pi] \to \mathbb{C}$ be a continuously differentiable function which has $f(-\pi) = f(\pi)$. How do the Fourier coefficients of the derivative f' relate to the Fourier coefficients of f? Prove *Poincaré's inequality:* if

$$\int_{-\pi}^{\pi} f(x)\, dx = 0,$$

then

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leqslant \int_{-\pi}^{\pi} |f'(x)|^2\, dx.$$

When does equality hold?

(3) The Legendre polynomials may be defined by the formula

$$P_0(x) = 1, \quad P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2-1)^n.$$

Show that the polynomials $\{\sqrt{n+\frac{1}{2}}\, P_n\}$ form an orthonormal set in the Hilbert space $L^2[-1,1]$. Show further that this orthonormal set is complete. (You may assume the Weierstrass approximation theorem, that every continuous function on $[-1,1]$ can be uniformly approximated by polynomials.)