Math 501 Homework 1 Due Friday, September 12th

(1) Let $f = u + iv$ be a holomorphic function of $z = x + iy$. Show that the function $g = \log |f|^2 = \log(u^2 + v^2)$ satisfies Laplace's equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

(You may assume that u and v are twice continuously differentiable.)

(2) Let z be a complex 5th root of 1, $z \neq 1$. Show that

$$1 + z + z^2 + z^3 + z^4 = 0.$$

By taking real parts deduce that

$$1 + 2\cos(2\pi/5) + 2\cos(4\pi/5) = 0,$$

and hence show that

$$\cos(2\pi/5) = \frac{\sqrt{5} - 1}{4}.$$

(3) Let z be a complex number with positive real part. By induction on n, show that

$$\int_0^1 t^{z-1}(1-t)^n \, dt = \frac{n!}{z(z+1)\cdots(z+n)}.$$

Substitute $t = u/n$ and let $n \to \infty$ to obtain

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^{z-1}}{z(z+1)\cdots(z+n-1)},$$

where Γ is the Gamma function defined in class. (Do your best to justify the limit processes that are used in this argument.)