Math 501 Homework 9 Solutions

(1) *Let V and W be normed vector spaces. Show that a linear map $T\colon V \to W$ is continuous if and only if it is* bounded, *which means that there is a constant c with*

$$\|T(v)\| \leqslant c\|v\|$$

for every $v \in V$.

If T is bounded, then from linearity we find

$$\|T(v) - T(w)\| \leqslant c\|v - w\|$$

for all vectors v and w, which certainly implies that T is continuous.

Conversely, if T is continuous then (by continuity at 0) there is a constant $\delta > 0$ such that, if $\|v\| \leqslant \delta$, then $\|T(v)\| \leqslant 1$. Put $c = 1/\delta$. Then for any $v \neq 0$, $u = v/(c\|v\|)$ has norm δ, and so $\|T(u)\| \leqslant 1$. By linearity of T it follows that $\|T(v)\| \leqslant c\|v\|$.

Let T be continuous and let $U = \ker T$. Show that U is a closed subspace of V and that the expression

$$\|[v]\| = \inf\{\|u + v\| : u \in U\}$$

where $[v]$ denotes the coset $v+U$, considered as an element of the quotient space V/U) defines a norm on V/U.

U is the inverse image of the closed set $\{0\}$ under the continuous map T, so it is closed. Now we want to show that the expression above defines a norm. It is easy to see that the expression is positive-linear and satisfies the triangle inequality. The only thing that we need to check is that if $\|[v]\| = 0$, then the coset $[v]$ is the zero element of the quotient space V/U, i.e. $v \in U$.

Since U is closed, its complement is open. Thus if $v \notin U$, there is a ball $B(v;\delta)$, for some $\delta > 0$, that does not meet U. That is to say, $\|u + v\| \geqslant \delta$ for all $u \in U$. It follows that

$$\|[v]\| = \inf\{\|u + v\| : u \in U\} \geqslant \delta > 0.$$

Contraposing, we find that if $\|[v]\| = 0$, then $v \in U$.

Show further that the map $[v] \mapsto T(v)$ is a well-defined continuous linear map from V/U to W.

This is a 'normed' version of the homomorphism theorem of linear algebra. To check that the map is well-defined one has to show that if $[v]$ and $[v']$ are the same coset, then $T(v) = T(v')$. This follows because if $[v] = [v']$ then $v - v' \in U = \ker T$. It is linear because T is linear. Finally, since

$$T(v) = T(v + u) \quad \forall\, u \in U,$$

we have $\|T(v)\| \leqslant c\|[v]\|$.

(2) *Let V be a (complex) normed vector space. A linear map $\varphi\colon V \to \mathbb{C}$ is called a* linear functional *on V. Show that a linear functional φ is continuous if and only if $\ker\varphi$ is closed. In fact, show that*

$$|\varphi(v)| \leqslant \frac{1}{\varepsilon}\|v\|,$$

where ε is the distance from $\ker\varphi$ to a vector v_1 with $\varphi(v_1) = 1$.

The implication φ continuous $\Rightarrow \ker\varphi$ closed follows from the previous question.

Conversely, suppose that $U = \ker\varphi$ is closed. If $\varphi \neq 0$ then there exist vectors v such that $\varphi(v) \neq 0$; by rescaling such a vector we can find a v_1 such that $\varphi(v_1) = 1$. Since U is closed there is a positive constant ε such that $\|v_1 + u\| \geqslant \varepsilon$ for all $u \in U$. (See the second part of the solution to the question before.)

Suppose now that $v \in V$ with $\varphi(v) \neq 0$. Put $u = v/\varphi(v) - v_1$. Then by calculation, $\varphi(u) = 0$, so $u \in U$. Since $v/\varphi(v) = v_1 + u$, we have $\|v/\varphi(v)\| \geqslant \varepsilon$. Rewriting this gives

$$|\varphi(v)| \leqslant \frac{1}{\varepsilon}\|v\|,$$

and this trivially also holds for $\varphi(v) = 0$. Thus φ is bounded, and therefore continuous.

Deduce, by induction on the dimension n, that if V is a finite-dimensional subspace of a normed vector space W, then V is closed in W and the linear isomorphism $V \to \mathbb{C}^n$ given by choosing a basis $\{v_1, \ldots, v_n\}$ of V is also a homeomorphism.

Let n be the dimension of V.

The case $n = 1$ is simple. The map $\lambda \mapsto \lambda v$, where v is a fixed element of V of norm 1, is a linear isometry $\mathbb{C} \to V$. Since $\mathbb{C}$ is complete, so is V. A subset of a metric space that is complete (considered as a metric space in its own right) is necessarily closed.

Now to the inductive step. Suppose V has a basis $\{v_1, \ldots, v_n\}$. For each k between 1 and n, consider the linear functional φ_k on V defined by

$$\varphi_k(\lambda_1 v_1 + \cdots \lambda_n v_n) = \lambda_k.$$

The kernel is $(n-1)$-dimensional, hence closed (by induction), so φ_k is continuous. Thus the linear isomorphism

$$\Phi = \bigoplus_k \varphi_k \colon V \to \mathbb{C}^n$$

is continuous. Its inverse $(\lambda_1, \dots, \lambda_n) \mapsto \sum \lambda_i v_i$ is continuous by construction. Thus Φ is a homeomorphism.

Since continuous linear maps are bounded there is a constant $C > 0$ such that if $v = \lambda_1 v_1 + \cdots \lambda_n v_n$ then

$$C^{-1}\|v\| \leqslant |\lambda_1| + \cdots + |\lambda_n| \leqslant C\|v\|.$$

This shows that Cauchy sequences in V map under Φ to Cauchy sequences in $\mathbb{C}^n$, and since the latter space is complete, so is the former. Finally, complete subspaces are closed (as we already remarked above) so V is closed. This completes the inductive step and therefore the proof.

(3) *Let V be a normed vector space. Suppose that the closed unit ball of V can be covered by finitely many balls $B(x_k; \frac{1}{2})$, $k = 1, \dots, n$, of radius $\frac{1}{2}$. Show that the $\{x_k\}$ span V.*

The span of $\{x_1, \cdots, x_n\}$ is a finite-dimensional subspace U of V, which is therefore closed (by the result of question 2). Suppose that $U \neq V$. Then for any vector $v \in V \setminus U$ the quantity $\inf\{\|v+u\| : u \in U\}$ is positive (as we saw above). By rescaling, we can find a vector v for which this quantity is exactly $\frac{3}{4}$.

By definition of the infimum there is $u \in U$ with $\|v+u\| < 1$. Let $w = v + u$. Then w belongs to the unit ball of V but its distance from any point of U — and in particular from any one of the $\{x_1, \dots, x_n\}$ — is at least $\frac{3}{4}$. So the union of the balls $B(x_k; \frac{1}{2})$ does not cover w, a contradiction. It follows that $U = V$, as required.

Deduce that every locally compact normed vector space is finite-dimensional.

If V is locally compact then some closed ball in V is compact. But all closed balls in a normed vector space are homeomorphic (by translation and rescaling), so the closed unit ball is compact. Thus it can be covered by finitely many open balls of radius $\frac{1}{2}$ and the previous argument applies.