List of Possible Research Projects for Math 497C
MASS Geometry and Topology
Preliminary Version

- Minimal surfaces.

  Discussion: A classical topic in differential geometry. A minimal surface is one whose *mean curvature* (different from the Gaussian curvature) equals zero. These are the shapes naturally taken up by 'soap films' — elastic surfaces trying to minimize their strain energy. Recent computer-aided progress by Hoffman, Meeks and others has resulted in the discovery of many new classes of minimal surfaces.

- Isoperimetric inequalities.

  Discussion: The prototype of an 'isoperimetric inequality' is the statement that a simple closed curve of perimeter p can enclose an area at most $p^2/4\pi$, and that maximum is attained only when the curve is a circle. This is one of the oldest theorems in differential geometry (e.g. see the legend of Dido's founding of Carthage, reported in the *Aeneid*). There are many proofs and many generalizations, for instance those involving the notion of 'Minkowski mixed volumes'. The isoperimetric inequality is also closely related to the *Sobolev embedding theorems* that occur in the study of partial differential equations.

  References: *What Is Mathematics?* (Courant and Hilbert); *Elementary Geometry* (Roe); *Geometry I and II* (Marcel Berger) provide sections discussing several different approaches to the isoperimetric inequality.

- Local Structure of Surfaces of Constant Curvature.

  Discussion: A surface of zero Gaussian curvature is locally isometric to a plane. A surface of constant curvature $+1$ is locally isometric to a sphere (of appropriate radius) and a surface of constant curvature -1 is locally isometric to hyperbolic space. One can also go on to ask what an *embedded* surface of constant curvature looks like. The example of a cylinder in 3-space shows for instance that one can have embedded surfaces of zero curvature that are not ambient isometric, even locally, to a flat embedded plane.

  References: Manfredo do Carmo, *Differential Geometry of Curves and Surfaces*, sections 4.6 and 5.8; Joseph Wolf, *Spaces of Constant Curvature*.

- The twin paradox: discussion and interpretation.

  See

  `math.ucr.edu/home/baez/physics/Relativity/SR/TwinParadox/twin_pa`

- Hilbert's theorem on non-immersability of the hyperbolic plane in 3-dimensional space.

  Discussion: Hilbert proved that any surface of constant negative curvature embedded in 3-dimensional Euclidean space must 'wrap up' like the tractoid. In other words, it is impossible to embed the hyperbolic plane (a complete, simply-connected surface of curvature -1) isometrically in 3-dimensional space.

  Reference: M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, section 5–11.

- Journey to the center of a black hole.

  Discussion: Suppose that you fall into a black hole of mass, say, a million times the mass of the Sun. What will you experience on your one-way journey to the central singularity? What will you see? What do you expect?

  Reference: *Exploring Black Holes: Introduction to General Relativity* by E.F. Taylor and J.A. Wheeler.

- Flexible polyhedra.

  Suppose that a polyhedron is made with rigid faces (cut out of card, say) that are hinged at the edges; can it be 'flexible' (that is, continuously deformable). Cauchy proved in the 19th century that the answer is 'no' for convex polyhedra. However, rather recently, examples of non-convex flexible polyhedra have been found.

  Reference: M. Berger, *Geometry II*, section 12.8, and references contained therein.

- The Yang-Mills equations and topology.

  Discussion: The Yang-Mills equations are certain geometric equations relating to curvature — associated, not now to a metric tensor, but to an abstract 'connection' on a vector bundle. The equations state that the curvature should have a certain symmetry property which is peculiar to 4-dimensional space. They were proposed originally in high energy physics, but in the early 1980s it was discovered that they can be used to solve outstanding problems in 4-dimensional topology, including the construction of an “exotic 4-space”; a space which is topologically but not smoothly equivalent to standard 4-dimensional Euclidean space.

References: The book by Donaldson and Kronheimer is the standard, but it is heavy going. Try looking at Michael Atiyah’s citation for Donaldson’s Fields Medal, in the Proceedings of the 1986 International Congress of Mathematicians (Berkeley, California).

- Gravitational waves.

  Discuss the generation and possible detection of gravitational waves in general relativity. Make any approximations that you feel are appropriate. What are the significant differences between gravitational and electromagnetic radiation?

  Reference: Misner, Thorne and Wheeler, *Gravitation* (the big black book, aka the ‘telephone directory’), section VIII (pages 974–1044).

- Rotating black holes.

  Discuss the Kerr metric.

  Reference: *Exploring Black Holes: Introduction to General Relativity* by E.F. Taylor and J.A. Wheeler.

- The positive mass theorem.

  This is a famous recent result in general relativity. Roughly speaking, it states that the total ‘mass’ of the gravitational field (a form of energy, so in relativity it must have some mass) is positive. This is a significant result because the definition of this ‘total mass’ is highly indirect.

  On reflection I think this one is too hard. But if you are not deterred you might look in the book ‘Black Hole Uniqueness Theorems’ by Heusler (Cambridge University Press).

- Gauss-Bonnet theorem and its generalizations.

  The Gauss-Bonnet theorem states that the total curvature (the integral of the Gaussian curvature) of a closed surface is 2π times the Euler characteristic ($V - E + F$, where V is the number of vertices, E the number of edges, and

F the number of faces). This beautiful and far-reaching theorem is proved in almost every book on surface geometry, including do Carmo's (which I have cited several times above), and my own 'Elementary Geometry'. See also *Riemannian Geometry: A Beginner's Guide* by Frank Morgan. The key ingredient is Green's theorem, or some equivalent version of 'integration by parts'. If you wanted some additional challenge you could also consider the Chern-Allendoefer generalization from the 1940s. You will find this in Milnor and Stasheff's 'Characteristic Classes', appendix C. There is a lot of new technology which goes into the generalization from dimension two to dimension four. .

- Gromov's notion of hyperbolicity for metric spaces.

  We are (going to) study the geometry of *hyperbolic space*, where triangles are much 'thinner' than they are in regular Euclidean geometry. In the early 1980's M. Gromov realized that this 'thin triangles' notion could be abstracted to provide a notion of 'large scale' hyperbolicity or tree-likeness for metric spaces. This is now a rich theory which is also elementary in a certain sense — at the beginning level it involves just clever manipulation of the triangle inequality. A trendy application is to claim that certain 'naturally arising' objects — the Internet is a favorite example — are in fact hyperbolic in Gromov's sense. I do not set much store by that myself, however.

  References: I like the book by Ghys and de la Harpe, 'Sur les groupes hyperboliques d'apres M. Gromov', but you need to read French.

- The Hilbert-Einstein action principle.

  Discussion: An 'action principle' in physics states that the universe likes to be in a configuration which maximizes (or minimizes, or at least renders stationary) some integral — the 'action'. For instance, classical mechanics can be viewed in these terms (Lagrange, Hamilton). Shortly after Einstein published the equations of general relativity it was realized by Hilbert that these equations can also be derived from an action principle; the action is the integral of the *scalar curvature* R_{ik}^{ik}.

  Refs: Einstein, *Hamilton's Principle and the General Theory of Relativity* (in the course text); Dirac, *General Relativity*; Misner, Thorne and Wheeler, chapter 21.

- Gravitational effects on the Global Positioning System.

  Reference: *Exploring Black Holes: Introduction to General Relativity* by E.F. Taylor and J.A. Wheeler.

- Topological classification of closed surfaces.

  Discuss the classification of closed surfaces (in 3-space, say) up to homeomorphism. It turns out that the 'number of holes' is a complete invariant. It takes some work to be able to see this.

  Reference: *Basic Topology* by M.A.Armstrong (Springer)

These are only a selection of possible research projects. A student who wishes to propose a different project is welcome to do so.