Math 497C Homework 5 — Due October 10th

(1) Let b_{ij} and g_{ij} be real symmetric 2×2 matrices, with g_{ij} positive definite. Let μ and μ' be the maximum and minimum values of the quadratic form

$$b_{ij}\lambda^i\lambda^j,$$

where the λ^i vary subject to the constraint $g_{ij}\lambda^i\lambda^j = 1$.

Prove that $\mu\mu' = (\det b)/(\det g)$.

(2) Let $(x^1(s), x^2(s))$ be a curve in a surface Σ, parameterized by arc length. The *normal curvature* to the curve at a given point is the component of $d^2\mathbf{r}/ds^2$ in the direction of the unit normal **n**; in other words it is the dot product $(d^2\mathbf{r}/ds^2)\cdot\mathbf{n}$.

Show that the normal curvature is equal to

$$b_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds},$$

where $b_{ij} = \mathbf{r}_{ij}\cdot\mathbf{n}$.

The greatest and least values of the normal curvature at a point are called the *principal curvatures*. Show that the product of the principal curvatures at a given point is equal to the Gaussian curvature at that point. (Use the previous question.)

(3) Find the Gaussian curvature of the tractoid (see HW 3).